Warning's Second Theorem with Restricted Variables

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Joint work with Pete L. Clark (U. Georgia) and Aden Forrow (M.I.T.)

A puzzle without a unique solution

							1	5
7	9							
			2					
					8	7		6
		1						
						9		
	7					8	3	
			1	5				
			3					

Figure: A 16-clue Sudoku puzzle

Outline of this talk

1. Zeros of polynomial systems
   - Artin's conjecture
   - Chevalley-Warning Theorem
   - Warning's Second Theorem
2. Tools from the polynomial method
   - Alon's Combinatorial Nullstellensatz
   - Schauz's generalization
   - Alon-Füredi Theorem
3. Warning's Second Theorem
   - A short(!) proof via Alon-Füredi Theorem
   - A generalization
   - Applications

Conjecture

(Artin's Conjecture) Let $n, d \in \mathbb{Z}^+$ with

$$d < n.$$

Let $P_1(t_1, \ldots, t_n) \in \mathbb{F}_q[t_1, \ldots, t_n]$ be a homogeneous polynomial of degree d. Let

$$Z = Z(P_1) = \{x \in \mathbb{F}_q^n \mid P_1(x) = 0\}$$

be the zero set in $\mathbb{F}_q^n$ of P_1, and let $\mathbf{z} = \#Z$. Then we have $\mathbf{z} \geq 2$.

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Artin was considering Wedderburn's celebrated theorem that every finite division ring is a field.

Theorem

(Chevalley-   Theorem) Let $n, r, d_1, \ldots, d_r \in \mathbb{Z}^+$ with $d_1 + \ldots + d_r < n$. For $1 \leq i \leq r$, let $P_i(t_1, \ldots, t_n) \in \mathbb{F}_q[t_1, \ldots, t_n]$ be a polynomial of degree d_i. Let

$$Z = Z(P_1, \ldots, P_r) = \{x \in \mathbb{F}_q^n \mid P_1(x) = \ldots = P_r(x) = 0\}$$

be the common zero set in $\mathbb{F}_q^n$ of the P_i's, and let $\mathbf{z} = \#Z$. Then:
a) (Chevalley's Theorem, 1935) We have $\mathbf{z} = 0$ or $\mathbf{z} \geq 2$.

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Theorem

(Warning's Second Theorem) With same hypotheses,

$$\mathbf{z} = 0 \text{ or } \mathbf{z} \geq q^{n-d}.$$

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Lemma

Let $\mathbb{F}$ be an arbitrary field, and let $f = f(t)$ be a polynomial in $\mathbb{F}[t]$. Suppose the degree of f is α (thus the t^α coefficient of f is nonzero). Then, if A is a subset of $\mathbb{F}$ with $|A| > \alpha$, there is an $a \in A$ so that

$$f(a) \neq 0.$$

Combinatorial Nullstellensatz

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Theorem

[Combinatorial Nullstellensatz (Part 2), N. Alon 1999] Let $\mathbb{F}$ *be an arbitrary field, and let* $f = f(t_1, \ldots, t_n)$ *be a polynomial in* $\mathbb{F}[t_1, \ldots, t_n]$*. Suppose the degree* $\deg(f)$ *of* f *is* $\sum_{i=1}^{n} \alpha_i$*, where each* α_i *is a nonnegative integer, and suppose the coefficient of* $\prod_{i=1}^{n} t_i^{\alpha_i}$ *in* f *is nonzero. Then, if* $A_1, \ldots, A_n$ *are subsets of* $\mathbb{F}$ *with* $|A_i| > \alpha_i$*, there are* $a_1 \in A_1, \ldots, a_n \in A_n$ *so that* $f(a_1, \ldots, a_n) \neq 0$*.*

Applications:

- Chevalley's theorem,
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Some challenges:

- Finding an encoding polynomial,
- having its degree 'low', and
- computing an appropriate coefficient.

Set-up for Schauz's Coefficient Formula

Given a polynomial $f \in \mathbb{F}[t_1, \ldots, t_n]$, define the *support of* f, $\mathrm{Supp}(f)$, as the set of all $(\alpha_1, \ldots, \alpha_n)$ such that the coefficient of $t_1^{\alpha_1} \ldots t_n^{\alpha_n}$ in f is nonzero. We say $(\alpha_1, \ldots, \alpha_n) \geq (\beta_1, \ldots, \beta_n)$ if $\alpha_i \geq \beta_i$ for all i; this gives us a partial ordering of the elements of $\mathrm{Supp}(f)$.

Schauz's Coefficient Formula - sharpening Alon's CN

Theorem

[Coefficient Formula, U. Schauz 2008]
Let f be a polynomial in $\mathbb{F}[t_1, \ldots, t_n]$ and let $f_{\alpha_1,\ldots,\alpha_n}$ denote the coefficient of $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ in f. Suppose that there is no greater element than $(\alpha_1, \ldots, \alpha_n)$ in Supp(f). Then for any sets $A_1, \ldots, A_n$ in $\mathbb{F}$ such that $|A_i| = \alpha_i + 1$ we have

$$f_{\alpha_1,\ldots,\alpha_n} = \sum_{(a_1,\ldots,a_n)\in A_1\times\cdots\times A_n} \frac{f(a_1, \ldots, a_n)}{N(a_1, \ldots, a_n)}, \tag{1}$$

where $N(a_1, \ldots, a_n) = \prod_{i=1}^{n} \prod_{b\in A_i\setminus\{a_i\}}(a_i - b)$.

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Note that this is 'backwards' to how we usually think – here we find coefficients from values, not values from the coefficients.

Corollary

[Schauz's Non-uniqueness Theorem, U. Schauz 2008] If $f_{\alpha_1,\ldots,\alpha_n} = 0$*, then either* f *vanishes over* $A_1 \times \cdots \times A_n$ *or* f *has at least two nonzeros over* $A_1 \times \cdots \times A_n$*.*

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Corollary

[U. Schauz 2008] Let $\mathbb{F}$ be an arbitrary field, and let f be a polynomial of degree d in $\mathbb{F}[t_1, \ldots, t_n]$. Then for any subsets $A_1, \ldots, A_n$ of $\mathbb{F}$ satisfying $\sum_{i=1}^{n}(|A_i| - 1) > d$, f either vanishes over $A_1 \times \cdots \times A_n$ or f has at least two nonzeros over $A_1 \times \cdots \times A_n$.

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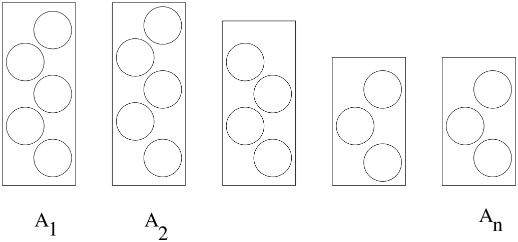
If the degree of the polynomial is small relative to the set we look over, then there cannot be a unique nonzero value.

Consequences

Using these ideas, Schauz gave proofs of:

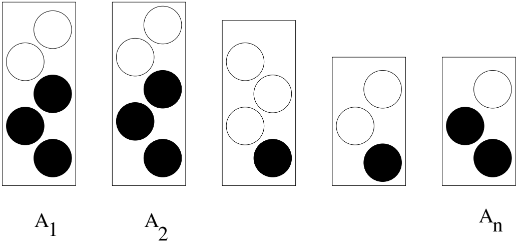
- Warning's Theorem
- a restricted variables Chevalley's Theorem

Balls in bins lemma



Bin A_i holds at most a_i balls.

Balls in bins lemma



Bin A_i holds at most a_i balls. Distribution of N balls is an n-tuple $y = (y_1, \ldots, y_n)$ with $y_1 + \ldots + y_n = N$ and $1 \leq y_i \leq a_i$ for all i.

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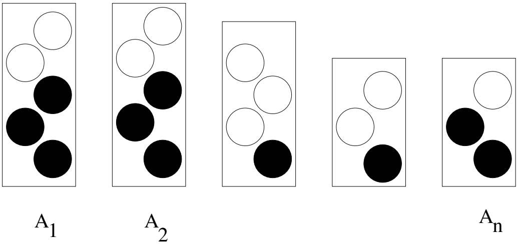

Let $\Pi(y) = y_1 \cdots y_n$. If $n \leq N \leq a_1 + \ldots + a_n$, let $\mathfrak{m}(a_1, \ldots, a_n; N)$ be the minimum value of $\Pi(y)$ as y ranges over all distributions of N balls into bins $A_1, \ldots, A_n$.

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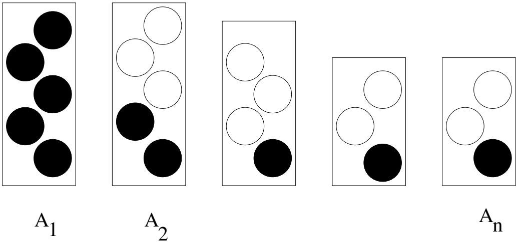

Let $P(y) = y_1 \cdots y_n$. If $n \leq N \leq a_1 + \ldots + a_n$, let $\mathfrak{m}(a_1, \ldots, a_n; N)$ be the minimum value of $\Pi(y)$ as y ranges over all distributions of N balls into bins $A_1, \ldots, A_n$. To minimize the product: serve the largest bins first.

Alon-Füredi Theorem

Theorem

(Alon-Füredi Theorem) Let $\mathbb{F}$ *be a field, let* $A_1, \ldots, A_n$ *be nonempty finite subsets of* $\mathbb{F}$*. Put* $A = \prod_{i=1}^{n} A_i$ *and* $a_i = \# A_i$ *for all* $1 \leq i \leq n$*. Let* $P \in \mathbb{F}[t] = \mathbb{F}[t_1, \ldots, t_n]$ *be a polynomial. Let*

$$\mathcal{U}_A = \{x \in A \mid P(x) \neq 0\}, \ \mathfrak{u}_A = \# \mathcal{U}_A.$$

Then $\mathfrak{u}_A = 0$ *or* $\mathfrak{u}_A \geq \mathfrak{m}(a_1, \ldots, a_n; a_1 + \ldots + a_n - \deg P)$*.*

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Proof.

Induction on n. □

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be the common zero set in $\mathbb{F}_q^n$ *of the* P_i*'s, and let* $\mathbf{z} = \#Z$*. Then:*

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Proof of Warning's Second Theorem via Alon-Füredi Theorem

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Applying the Alon-Füredi Theorem we get $\mathbf{z}_A = 0$ or

$$\mathbf{z}_A \geq \mathfrak{m}(\#A_1 + \ldots + \#A_n; \#A_1 + \ldots + \#A_n - (q-1)d).$$

Theorem

(Restricted Variable Warning's Second Theorem, P. Clark, A. Forrow, S. - 2014+) *Let K be a number field with ring of integers R, let $\mathfrak{p}$ be a nonzero prime ideal of R, and let $q = p^\ell$ be the prime power such that $R/\mathfrak{p} \cong \mathbb{F}_q$. Let $A_1, \ldots, A_n$ be nonempty subsets of R such that for each i, the elements of A_i are pairwise incongruent modulo $\mathfrak{p}$, and put $A = \prod_{i=1}^n A_i$. Let $r, v_1, \ldots, v_r \in \mathbb{Z}^+$. Let $P_1, \ldots, P_r \in R[t_1, \ldots, t_n]$. Let*

$$Z_A = \{x \in A \mid P_j(x) \equiv 0 \pmod{\mathfrak{p}^{v_j}} \ \forall 1 \leq j \leq r\}, \ \mathbf{z}_A = \# Z_A.$$

a) $\mathbf{z}_A = 0$ *or* $\mathbf{z}_A \geq \mathfrak{m}\left(\#A_1, \ldots, \#A_n; \#A_1 + \ldots + \#A_n - \sum_{j=1}^r (q^{v_j} - 1)\deg(P_j)\right)$.

*b) (***Boolean Case***) We have* $\mathbf{z}_{\{0,1\}^n} = 0$ *or* $\mathbf{z}_{\{0,1\}^n} \geq 2^{n - \sum_{j=1}^r (q^{v_j} - 1)\deg(P_j)}$.

The theorem recovers:

- Warning's Second Theorem
- Schanuel's Theorem (reproved by Baker-Schmidt) for polynomial systems over the rings $\mathbb{Z}/p^{v_j}\mathbb{Z}$
- Schauz's restricted variable Chevalley Theorem
- Schauz's (and later Brink's) generalization of these for polynomial systems over $\mathbb{Z}$

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$$6 + 1 + 11 + 1 + 16 = 35$$

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- we win!
The pigeonhole principle applied to the partial sums shows that n draws is enough.

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So, $2n - 2$ draws is not enough. Perhaps $2n - 1$ is?

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$$P_1(t_1, \ldots, t_m) = \sum_{i=1}^{m} b_i t_i \in \mathbb{F}_p[t_1, \ldots, t_m]$$

and

$$P_2(t_1, \ldots t_m) = \sum_{i=1}^{m} t_i \in \mathbb{F}_p[t_1, \ldots, t_m].$$

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P_1 encodes divisibility condition on sum. P_2 encodes number of terms in sequence.
$deg(P_1) + deg(P_2) = 1 + 1 = 2$ and
$P_1(0, \ldots, 0) = P_2(0, \ldots, 0) = 0$.
Restrict to Boolean case of RVW2T: get

$$\mathbf{z}_{\{0,1\}^n} \geq 2^{m-2(p-1)}.$$

Thus, when $m > 2p - 2$ we have non-trivial solutions.

Chevalley's Theorem $\implies$ Erdős-Ginzburg-Ziv
Schanuel's Theorem: computes Davenport constant of finite commutative p-groups
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Restricted Variable Warning's Second Theorem:
applies to each of above to get quantitative refinements, which include inhomogeneous case;
tool to refine combinatorial existence theorems into theorems which give explicit lower bounds on number of combinatorial objects asserted to exist.

Thank you.