Warning’s Second Theorem with Restricted Variables

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Joint work with Pete L. Clark (U. Georgia) and Aden Forrow (M.I.T.)
A puzzle without a unique solution

![A 16-clue Sudoku puzzle](image)

**Figure:** A 16-clue Sudoku puzzle
Outline of this talk

1. Zeros of polynomial systems
   - Artin’s conjecture
   - Chevalley-Warning Theorem
   - Warning’s Second Theorem

2. Tools from the polynomial method
   - Alon’s Combinatorial Nullstellensatz
   - Schauz’s generalization
   - Alon-Füredi Theorem

3. Warning’s Second Theorem
   - A short(!) proof via Alon-Füredi Theorem
   - A generalization
   - Applications
Conjecture

(Artin’s Conjecture) Let $n, d \in \mathbb{Z}^+$ with $d < n$.

Let $P_1(t_1, \ldots, t_n) \in \mathbb{F}_q[t_1, \ldots, t_n]$ be a homogeneous polynomial of degree $d$. Let

$$Z = Z(P_1) = \{x \in \mathbb{F}_q^n \mid P_1(x) = 0\}$$

be the zero set in $\mathbb{F}_q^n$ of $P_1$, and let $z = \#Z$. Then we have $z \geq 2$. 
Conjecture

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Artin was considering Wedderburn’s celebrated theorem that every finite division ring is a field.
<table>
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| **(Chevalley-Theorem)** Let \( n, r, d_1, \ldots, d_r \in \mathbb{Z}^+ \) with \( d_1 + \ldots + d_r < n \). For \( 1 \leq i \leq r \), let \( P_i(t_1, \ldots, t_n) \in \mathbb{F}_q[t_1, \ldots, t_n] \) be a polynomial of degree \( d_i \). Let \[
Z = Z(P_1, \ldots, P_r) = \{x \in \mathbb{F}_q^n \mid P_1(x) = \ldots = P_r(x) = 0\}
\]
be the common zero set in \( \mathbb{F}_q^n \) of the \( P_i \)'s, and let \( z = \#Z \). Then:

a) **(Chevalley’s Theorem, 1935)** We have \( z = 0 \) or \( z \geq 2 \).
### Theorem

*(Chevalley-Warning Theorem)* Let $n, r, d_1, \ldots, d_r \in \mathbb{Z}^+$ with $d_1 + \ldots + d_r < n$. For $1 \leq i \leq r$, let $P_i(t_1, \ldots, t_n) \in \mathbb{F}_q[t_1, \ldots, t_n]$ be a polynomial of degree $d_i$. Let

$$Z = Z(P_1, \ldots, P_r) = \{x \in \mathbb{F}_q^n \mid P_1(x) = \ldots = P_r(x) = 0\}$$

be the common zero set in $\mathbb{F}_q^n$ of the $P_i$’s, and let $z = \#Z$. Then:

a) *(Chevalley’s Theorem, 1935)* We have $z = 0$ or $z \geq 2$.

b) *(Warning’s Theorem, 1935)* We have $z \equiv 0 \pmod{p}$. 

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**Warning’s Second Theorem with Restricted Variables**

Zeros of polynomial systems
Theorem

(Chevalley–Warning Theorem) Let $n, r, d_1, \ldots, d_r \in \mathbb{Z}^+$ with $d_1 + \ldots + d_r < n$. For $1 \leq i \leq r$, let $P_i(t_1, \ldots, t_n) \in \mathbb{F}_q[t_1, \ldots, t_n]$ be a polynomial of degree $d_i$. Let

$$Z = Z(P_1, \ldots, P_r) = \{x \in \mathbb{F}_q^n \mid P_1(x) = \ldots = P_r(x) = 0\}$$

be the common zero set in $\mathbb{F}_q^n$ of the $P_i$’s, and let $z = \#Z$. Then:

a) (Chevalley’s Theorem, 1935) We have $z = 0$ or $z \geq 2$.

b) (Warning’s Theorem, 1935) We have $z \equiv 0 \pmod{p}$.

Theorem

(Warning’s Second Theorem) With same hypotheses,

$$z = 0 \text{ or } z \geq q^{n-d}.$$
The polynomial method!

Encode combinatorial problems via a polynomial so that nonzeros of polynomial correspond to solutions of the combinatorial problem.
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\[ P(t) = \prod_{i=1}^{r} (1 - P_i(t)^{q-1}) \]
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Encode combinatorial problems via a polynomial so that nonzeros of polynomial correspond to solutions of the combinatorial problem.

\[ P(t) = \prod_{i=1}^{r} (1 - P_i(t)^{q-1}) \]

\( P(t) \) is zero whenever any \( P_i \) is nonzero.
\( P(t) \) is nonzero only when each \( P_i \) is zero.
A basic theorem of algebra

**Fact:** A one variable polynomial over a field $\mathbb{F}$ can have at most as many zeros as its degree.
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**Example:** $f(t) = t^2 - 1 \in \mathbb{R}[t]$ and the set $A = \{1, -1, 3\}$. $f(3) \neq 0$
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**Fact:** A one variable polynomial over a field $\mathbb{F}$ can have at most as many zeros as its degree.

**Example:** $f(t) = t^2 - 1 \in \mathbb{R}[t]$ and the set $A = \{1, -1, 3\}$. $f(3) \neq 0$

**Lemma**

Let $\mathbb{F}$ be an arbitrary field, and let $f = f(t)$ be a polynomial in $\mathbb{F}[t]$. Suppose the degree of $f$ is $\alpha$ (thus the $t^\alpha$ coefficient of $f$ is nonzero). Then, if $A$ is a subset of $\mathbb{F}$ with $|A| > \alpha$, there is an $a \in A$ so that $f(a) \neq 0$. 
A ‘low’ degree polynomial evaluated over a ‘large’ box has a nonzero.
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Theorem

[Combinatorial Nullstellensatz (Part 2), N. Alon 1999] Let $\mathbb{F}$ be an arbitrary field, and let $f = f(t_1, \ldots, t_n)$ be a polynomial in $\mathbb{F}[t_1, \ldots, t_n]$. Suppose the degree $\deg(f)$ of $f$ is $\sum_{i=1}^{n} \alpha_i$, where each $\alpha_i$ is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^{n} t_i^{\alpha_i}$ in $f$ is nonzero. Then, if $A_1, \ldots, A_n$ are subsets of $\mathbb{F}$ with $|A_i| > \alpha_i$, there are $a_1 \in A_1, \ldots, a_n \in A_n$ so that $f(a_1, \ldots, a_n) \neq 0$. 
Applications:
- Chevalley’s theorem,
- graph coloring,
- the Permanent Lemma,
- and many, many more.
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- Chevalley’s theorem,
- graph coloring,
- the Permanent Lemma,
- and many, many more.

Some challenges:

- Finding an encoding polynomial,
- having its degree ‘low’, and
- computing an appropriate coefficient.
Set-up for Schauz’s Coefficient Formula

Given a polynomial $f \in \mathbb{F}[t_1, \ldots, t_n]$, define the support of $f$, \text{Supp}(f), as the set of all $(\alpha_1, \ldots, \alpha_n)$ such that the coefficient of $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ in $f$ is nonzero. We say $(\alpha_1, \ldots, \alpha_n) \geq (\beta_1, \ldots, \beta_n)$ if $\alpha_i \geq \beta_i$ for all $i$; this gives us a partial ordering of the elements of \text{Supp}(f).
Schauz’s Coefficient Formula - sharpening Alon’s CN

Theorem

[Coefficient Formula, U. Schauz 2008]
Let $f$ be a polynomial in $\mathbb{F}[t_1, \ldots, t_n]$ and let $f_{\alpha_1, \ldots, \alpha_n}$ denote the coefficient of $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ in $f$. Suppose that there is no greater element than $(\alpha_1, \ldots, \alpha_n)$ in $\text{Supp}(f)$. Then for any sets $A_1, \ldots, A_n$ in $\mathbb{F}$ such that $|A_i| = \alpha_i + 1$ we have

$$f_{\alpha_1, \ldots, \alpha_n} = \sum_{(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n} \frac{f(a_1, \ldots, a_n)}{N(a_1, \ldots, a_n)},$$

(1)

where $N(a_1, \ldots, a_n) = \prod_{i=1}^n \prod_{b \in A_i \setminus \{a_i\}} (a_i - b)$. 
Schauz’s Coefficient Formula - sharpening Alon’s CN

**Theorem**

[Coefficient Formula, U. Schauz 2008]

Let \( f \) be a polynomial in \( \mathbb{F}[t_1, \ldots, t_n] \) and let \( f_{\alpha_1, \ldots, \alpha_n} \) denote the coefficient of \( t_1^{\alpha_1} \cdots t_n^{\alpha_n} \) in \( f \). Suppose that there is no greater element than \( (\alpha_1, \ldots, \alpha_n) \) in \( \text{Supp}(f) \). Then for any sets \( A_1, \ldots, A_n \) in \( \mathbb{F} \) such that \( |A_i| = \alpha_i + 1 \) we have

\[
f_{\alpha_1, \ldots, \alpha_n} = \sum_{(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n} \frac{f(a_1, \ldots, a_n)}{N(a_1, \ldots, a_n)}, \tag{1}
\]

where \( N(a_1, \ldots, a_n) = \prod_{i=1}^n \prod_{b \in A_i \setminus \{a_i\}} (a_i - b) \).

Note that this is ‘backwards’ to how we usually think – here we find coefficients from values, not values from the coefficients.
Corollary

[Schauz’s Non-uniqueness Theorem, U. Schauz 2008] If \( f_{\alpha_1, \ldots, \alpha_n} = 0 \), then either \( f \) vanishes over \( A_1 \times \cdots \times A_n \) or \( f \) has at least two nonzeros over \( A_1 \times \cdots \times A_n \).
Corollary

[Schauz’s Non-uniqueness Theorem, U. Schauz 2008] If $f_{\alpha_1, \ldots, \alpha_n} = 0$, then either $f$ vanishes over $A_1 \times \cdots \times A_n$ or $f$ has at least two nonzeros over $A_1 \times \cdots \times A_n$.

Corollary

[U. Schauz 2008] Let $\mathbb{F}$ be an arbitrary field, and let $f$ be a polynomial of degree $d$ in $\mathbb{F}[t_1, \ldots, t_n]$. Then for any subsets $A_1, \ldots, A_n$ of $\mathbb{F}$ satisfying $\sum_{i=1}^n (|A_i| - 1) > d$, $f$ either vanishes over $A_1 \times \cdots \times A_n$ or $f$ has at least two nonzeros over $A_1 \times \cdots \times A_n$. 
Corollary

[Schauz's Non-uniqueness Theorem, U. Schauz 2008] If $f_{\alpha_1,\ldots,\alpha_n} = 0$, then either $f$ vanishes over $A_1 \times \cdots \times A_n$ or $f$ has at least two nonzeros over $A_1 \times \cdots \times A_n$.

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If the degree of the polynomial is small relative to the set we look over, then there cannot be a unique nonzero value.
Consequences

Using these ideas, Schauz gave proofs of:

- Warning’s Theorem
- a restricted variables Chevalley’s Theorem
Balls in bins lemma

Bin $A_i$ holds at most $a_i$ balls.
Balls in bins lemma

Bin $A_i$ holds at most $a_i$ balls. Distribution of $N$ balls is an $n$-tuple $y = (y_1, \ldots, y_n)$ with $y_1 + \ldots + y_n = N$ and $1 \leq y_i \leq a_i$ for all $i$. 
Let $\Pi(y) = y_1 \cdots y_n$. If $n \leq N \leq a_1 + \ldots + a_n$, let $m(a_1, \ldots, a_n; N)$ be the minimum value of $\Pi(y)$ as $y$ ranges over all distributions of $N$ balls into bins $A_1, \ldots, A_n$. 
Let $P(y) = y_1 \cdots y_n$. If $n \leq N \leq a_1 + \ldots + a_n$, let $m(a_1, \ldots, a_n; N)$ be the minimum value of $\prod(y)$ as $y$ ranges over all distributions of $N$ balls into bins $A_1, \ldots, A_n$. To minimize the product: serve the largest bins first.
Alon-Füredi Theorem

**Theorem**

(Alon-Füredi Theorem) Let $\mathbb{F}$ be a field, let $A_1, \ldots, A_n$ be nonempty finite subsets of $\mathbb{F}$. Put $A = \prod_{i=1}^{n} A_i$ and $a_i = \#A_i$ for all $1 \leq i \leq n$. Let $P \in \mathbb{F}[t] = \mathbb{F}[t_1, \ldots, t_n]$ be a polynomial. Let

$$U_A = \{x \in A \mid P(x) \neq 0\}, \; u_A = \#U_A.$$

Then $u_A = 0$ or $u_A \geq m(a_1, \ldots, a_n; a_1 + \ldots + a_n - \deg P)$. 
(Alon-Füredi Theorem) Let $\mathbb{F}$ be a field, let $A_1, \ldots, A_n$ be nonempty finite subsets of $\mathbb{F}$. Put $A = \prod_{i=1}^{n} A_i$ and $a_i = \#A_i$ for all $1 \leq i \leq n$. Let $P \in \mathbb{F}[t] = \mathbb{F}[t_1, \ldots, t_n]$ be a polynomial. Let

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Then $u_A = 0$ or $u_A \geq m(a_1, \ldots, a_n; a_1 + \ldots + a_n - \deg P)$.

**Proof.**

Induction on $n$. □
Warning’s Second Theorem

**Theorem**

Let $n, r, d_1, \ldots, d_r \in \mathbb{Z}^+$ with

$$d_1 + \ldots + d_r < n.$$ 

For $1 \leq i \leq r$, let $P_i(t_1, \ldots, t_n) \in \mathbb{F}_q[t_1, \ldots, t_n]$ be a polynomial of degree $d_i$. Let

$$Z = Z(P_1, \ldots, P_r) = \{x \in \mathbb{F}_q^n \mid P_1(x) = \ldots = P_r(x) = 0\}$$

be the common zero set in $\mathbb{F}_q^n$ of the $P_i$’s, and let $z = \#Z$. Then:

$$z = 0 \text{ or } z \geq q^{n-d}.$$
Proof of Warning’s Second Theorem via Alon-Füredi Theorem

Put

\[ P(t) = \prod_{i=1}^{r} (1 - P_i(t)^{q-1}). \]
Proof of Warning’s Second Theorem via Alon-Füredi Theorem

Put

\[ P(t) = \prod_{i=1}^{r} (1 - P_i(t)^q - 1). \]

Then \( \deg P = (q - 1)(\deg(P_1) + \ldots + \deg(P_r)) \), and

\[ U_A = \{ x \in A \mid P(x) \neq 0 \} = Z_A, \]

so

\[ z_A = \#Z_A = \#U_A = u_A. \]
Proof of Warning’s Second Theorem via Alon-Füredi Theorem

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Applying the Alon-Füredi Theorem we get \( z_A = 0 \) or

\[ z_A \geq m(\#A_1 + \ldots + \#A_n; \#A_1 + \ldots + \#A_n - (q - 1)d). \]
Warning’s Second Theorem (Restricted Variable)

Let $K$ be a number field with ring of integers $R$, let $p$ be a nonzero prime ideal of $R$, and let $q = p^\ell$ be the prime power such that $R/p \cong \mathbb{F}_q$. Let $A_1, \ldots, A_n$ be nonempty subsets of $R$ such that for each $i$, the elements of $A_i$ are pairwise incongruent modulo $p$, and put $A = \prod_{i=1}^{n} A_i$. Let $r, v_1, \ldots, v_r \in \mathbb{Z}^+$. Let $P_1, \ldots, P_r \in R[t_1, \ldots, t_n]$. Let

$$Z_A = \{ x \in A \mid P_j(x) \equiv 0 \pmod{p^{v_j}} \, \forall 1 \leq j \leq r \}, \quad z_A = \#Z_A.$$

a) $z_A = 0$ or $z_A \geq m\left(\#A_1, \ldots, \#A_n; \#A_1 + \ldots + \#A_n - \sum_{j=1}^{r} (q^{v_j} - 1) \deg(P_j)\right)$.

b) (Boolean Case) We have $z_{\{0,1\}^n} = 0$ or $z_{\{0,1\}^n} \geq 2^{n - \sum_{j=1}^{r} (q^{v_j} - 1) \deg(P_j)}$. 

(Warning’s Second Theorem, P. Clark, A. Forrow, S. - 2014+)
The theorem recovers:

- Warning’s Second Theorem
- Schanuel’s Theorem (reproved by Baker-Schmidt) for polynomial systems over the rings $\mathbb{Z}/p^{\nu_j}\mathbb{Z}$
- Schauz’s restricted variable Chevalley Theorem
- Schauz’s (and later Brink’s) generalization of these for polynomial systems over $\mathbb{Z}$
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Let’s draw integers from a bag and seek a subsequence of these with sum divisible by $n$. How many draws must we take? Say $n = 5$ and we draw $b_1 = 6$, $b_2 = 1$, $b_3 = 11$, $b_4 = 1$, and $b_5 = 16$. The sum is $6 + 1 + 11 + 1 + 16 = 35$, so we win! By the pigeonhole principle applied to the partial sums, $n$ draws is enough.
Let’s draw integers from a bag and seek a subsequence of these with sum divisible by $n$. How many draws must we take? Say $n = 5$ and we draw $b_1 = 6$, $b_2 = 1$, $b_3 = 11$, $6 + 1 + 11 + 1 + 16 = 35$ — we win! The pigeonhole principle applied to the partial sums shows that $n$ draws is enough.
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(Erdős-Ginzburg-Ziv 1961) Let’s draw integers from a bag and seek a subsequence of these with sum divisible by \( n \) and with the number of terms equal to \( n \). How many draws must we take?
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Say \( n = 5 \) and \( b_1 = 0, b_2 = 0, b_3 = 0, b_4 = 0 \) and \( b_5 = 1, b_6 = 1, b_7 = 1, b_8 = 1 \). So, \( 2n - 2 \) draws is not enough. Perhaps \( 2n - 1 \) is?
Given $n = p$ a prime, we sketch a proof that $2p - 1$ terms is enough. Let us consider a sequence of length $m$. 
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Let

$$P_1(t_1, \ldots, t_m) = \sum_{i=1}^{m} b_i t_i \in \mathbb{F}_p[t_1, \ldots, t_m]$$

and

$$P_2(t_1, \ldots t_m) = \sum_{i=1}^{m} t_i \in \mathbb{F}_p[t_1, \ldots, t_m].$$

$P_1$ encodes divisibility condition on sum. $P_2$ encodes number of terms in sequence.
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$P_1$ encodes divisibility condition on sum. $P_2$ encodes number of terms in sequence.

$\text{deg}(P_1) + \text{deg}(P_2) = 1 + 1 = 2$ and

$P_1(0, \ldots, 0) = P_2(0, \ldots, 0) = 0$.

Restrict to Boolean case of RVW2T: get

$$z_{\{0,1\}^n} \geq 2^{m-2(p-1)}.$$ 

Thus, when $m > 2p - 2$ we have non-trivial solutions.
Chevalley’s Theorem $\implies$ Erdős-Ginzburg-Ziv
Schanuel’s Theorem: computes Davenport constant of finite commutative $p$-groups
Schanuel’s Theorem: main technical input of result of Alon, Kleitman, Lipton, Meshulam, Rabin on selecting from set systems to get union of cardinality divisible by prime power $q$
Warning’s Second Theorem with Restricted Variables

Warning’s Second Theorem

Applications

Chevalley’s Theorem $\implies$ Erdős-Ginzburg-Ziv

Schanuel’s Theorem: computes Davenport constant of finite commutative $p$-groups

Schanuel’s Theorem: main technical input of result of Alon, Kleitman, Lipton, Meshulam, Rabin on selecting from set systems to get union of cardinality divisible by prime power $q$

Restricted Variable Warning’s Second Theorem: applies to each of above to get quantitative refinements, which include inhomogeneous case;

tool to refine combinatorial existence theorems into theorems which give explicit lower bounds on number of combinatorial objects asserted to exist.
Thank you.