Martin Gardner's Minimum No-Three-In-A-Line Problem

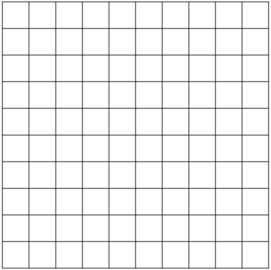
John Schmitt

Middlebury College, VT

joint work with Alec Cooper, Oleg Pikhurko, Greg Warrington

1. Give the problem
2. Give history and warm-up proof
3. Introduce Combinatorial Nullstellensatz and prove main result using it
4. (Almost) prove main result again, without resorting to algebra
5. Debate 'algebra' vs. 'combinatorics'

Consider a chessboard of size $n \times n$



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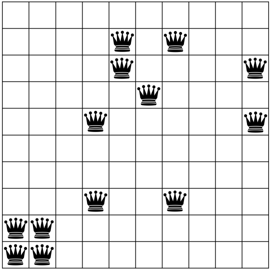

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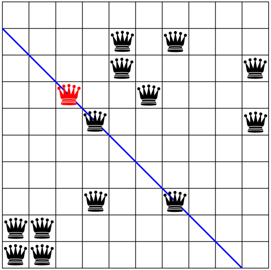

We want to place queens on it so that there are not three queens in a line, but the addition of one more queen would force there to be three in a line.

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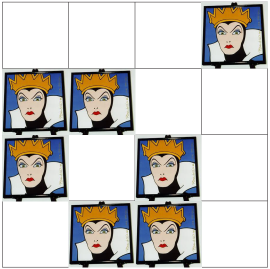

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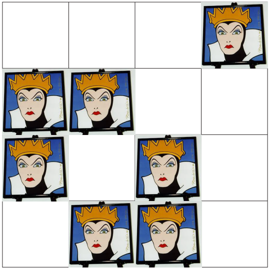
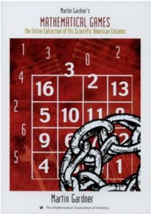


How few queens ($m_3(n)$) can we use and still satisfy these properties?

Martin Gardner



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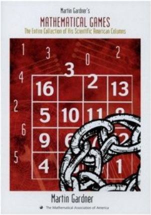

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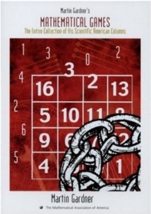

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Gardner first published the minimum no-three-in-line problem in the October 1976 issue of Scientific American.

Gardner makes the following observation,

> *If 'line' is taken in the broadest sense — a straight line of any orientation — the problem is difficult. . . The problem is also unsolved if 'line' is restricted to orthogonals and diagonals.*

Gardner corresponded with several people about this problem prior to the publication.

In particular, Gardner had received a proof from a man named John Harris (of Santa Barbara, CA) that at least n queens were always necessary to satisfy the conditions on an $n \times n$ board, except in the case that n is congruent to 3 modulo 4, in which case one less might be possible .

He mentioned the existence of this "proof" in his article, but did not give it.

Proposition

For all $n \geq 1$, $m_3(n) \geq \frac{n}{2}$.

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Proof.

Put q queens on the board.
Each queen occupies one square and sees at most $4n-4$ squares.
There are n^2 squares, each of which takes two queens to cover it or one queen to occupy it. So we have:

$$\begin{aligned} \frac{1}{2}(4n-4)q + q &\geq n^2 \\ (2n-1)q &\geq n^2 \\ 2nq &\geq n^2 \\ q &\geq \frac{n}{2} \end{aligned}$$

□

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Can be improved since only a few queens see $4n - 4$ squares though each sees at least $3n - 3$ squares, but at best this gets to a bound of $\frac{2}{3}n$.

Dear John: 2 June 75

Luckily you had sent me several pages, many years ago, about analysis of the heptominoes that tile and do not tile, so I've been able to summarize. You once said it was okay for me to use things mentioned in these early letters, so I've I called you procedure for checking the "Conway criterion," pictured the heptominoes to which it doesn't apply, identified the tilers, including the one you dubbed the "most interesting," and given as a problem for readers to finding of a tiling for one of the nontilers, combined with 3 x 3 squares. My other problem is finding a tiling for one of Penrose's poly- iamonds, that I call the loaded wheelbarrow (he gave me permission to use." It calls for 8 different orientations to form a fundamental region that tiles by translation.

I am postponing the column on "nonperiodic tiling" until next year, hoping that Penrose will give me permission by then to picture his two tiles. At that time I would very much wish to include your observations of this pattern -- and will do my best to alert you at least two months ahead of my deadline. So far, Penrose has no yet revealed to me the shapes of the tiles

Any chance of getting a look now at your two maps: the one for the Century puzzle, and for Dad's? I once did a column on sliding blocks, but one of these days I'll do another one, and I'd like to have these items on file with permission to use sometime. Does Knuth have them? I mention it because I think his next volume (on combinatoric algorithms) is going to include some sliding block material.

And OF COURSE I would like to get from you or somebody the new news about the angel problem!

I have no memory at this point of which of my column collections I've sent you, but there is a new paperback edition just out of the last one, The Sixth Book of etc., which contains my old column on sliding blocks. I have today put one in ship mail for you. Incidentally, I hope you don't mind that I have dedicated the seventh volume, Mathematical Carnival, to you, which is coming out this fall by Knopf. I have a few lines in the dedication, which I think you will like, but let me hold this off so there'll be some surprise left.

I enclose a piece about me in Time, occasioned by my April Fool hoax column, which produced about 2,000 letters from readers who didn't know it was a joke.

The current (June) column mentions Ulam's game of taking turns putting a counter on an n x n until one person wins by getting 3 in line, orthogonally or diagonally. This suggested to me the following problem, which I believe is new. What is the minimum number of counters that can be put on a square, no 3 in line, such that one more counter produces 3 in line?

I will enclose my best results through n = 12. I haven't been able to prove that my n = 8 (10 counters) is minimal, or to prove to at least n counters are required for all n, though I suspect it might not be hard to settle the latter conjecture one way or the other. It would be nice if the minimum for all odd n were N + 1, and for all even n, n or n + 2. If "line" is taken to any straight line on the field, the general problem becomes much more difficult. I note an article on the traditional "no-3-in-line" problem in the May J. of Comb. Theory, on which Guy has done some work. (I recently wrote him about all of this, asking for one of his papers on it, but haven't received a reply yet). He let me see a copy of his paper (forthcoming) on your Sylver coinage problem, about which I also have some comments from you in letters.

If you ever type up anything on Football, such as definitions of terms, etc., please keep me in mind for a copy. As I've said before, I'm holding off on this for a future column, for which SA would make a substantial payment (whether you like this or not), but anything you record about it I'd like to have on file, on a confidential basis. By fall, SA should know whether they've lost money of the six filmstrips I did (my work now completed, but strips not yet on market), which will determine how they will feel about trying a game.

I mistakenly told Penrose in a letter that the game agent here who handled Piet Hein, Tom Atwater, had retired from the game business. It is true that he quite his job with a lwa firm, to become dean of a business school, but I recently learned that he is continuing to act as agent for "select customers." Anyway, he can still be reached at his home, 15 Walden St., Concord, Mass 01742. In fact, I'll write him a letter today about Football, and ask him to contact you if he thinks it has market possibilities.

Best,

Dear Mr. Sands: 17 June 75

Many thanks for summarizing the results of that paper for me. I got lost on it at several spots.

Guy informs me that the reference to a solution for n = 12 is an error. On closer examination, the one solution that someone thought he found proved to have 3 in line! So -- no solutions, with 2n counters, are known beyond n = 10. It would be interesting, wouldn't it, if there are none higher than 1o?

I failed to keep a carbon of my letter, so I don't km recall if I mentioned that I have been working on the simpler problem of the minimum number of counters, no 3 in line, such that one more counter puts 3 in line, but defining "line" as a row, column or diagonal.

My minimums, 3 through 12, are:

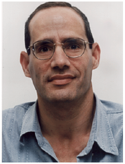
[illegible] 4,4,6,6,8,10,12,12,12.

I've been trying to prove that the minimum number of counters must be at least n, but the best I can do is prove it must be at least n - 1. (Actually, not my proof, but one from a correspondent, John Harris).

I can get 4-fold symmetry only for n = 4, 7 and 10,11, 12. The following pattern solves 10,11,12.

Our approach to the problem: Combinatorial Nullstellensatz

- Developed by Noga Alon in 1990s
- Leverages a special case of Hilbert's Nullstellensatz



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Lemma

Let $\mathbb{F}$ be an arbitrary field, and let $f = f(x)$ be a polynomial in $\mathbb{F}[x]$. Suppose the degree of f is t (thus the x^t coefficient of f is nonzero). Then, if S is a subset of $\mathbb{F}$ with $|S| > t$, there is an $s \in S$ so that

$$f(s) \neq 0.$$

Theorem (Part 2, Alon, 1999)

Let $\mathbb{F}$ be an arbitrary field, and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \ldots, x_n]$.
Suppose the degree $\deg(f)$ of f is $\sum_{i=1}^{n} t_i$, where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in f is nonzero.
Then, if $S_1, \ldots, S_n$ are subsets of $\mathbb{F}$ with $|S_i| > t_i$, there are $s_1 \in S_1, \ldots, s_n \in S_n$ so that

$$f(s_1, \ldots, s_n) \neq 0.$$

Lower Bound of n

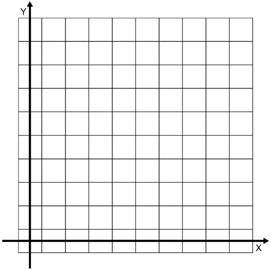
Theorem (Cooper, Pikhurko, S. , Warrington, '13)

For any $n \geq 1$, we have $m_3(n) \geq n$, except when $n \equiv 3 \pmod 4$ when $m_3(n) \geq n - 1$.

Lower Bound of n

Proof

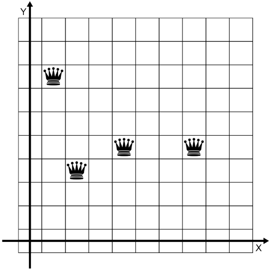
We restrict our attention to $n = 4k + 1$.



Place our chessboard into the standard Cartesian coordinate system.

Lower Bound of n

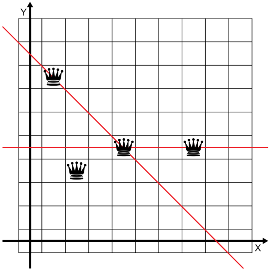
Proof



Suppose a set $\mathcal{Q}$ of $q \leq 4k$ queens have been placed, and that these queens satisfy the requirements.

Lower Bound of n

Proof



Then there are at most $2k$ lines in each direction defined by these queens.

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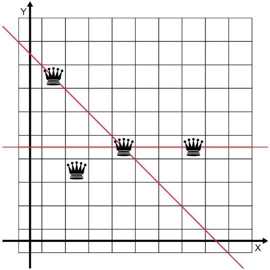
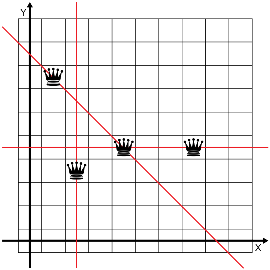


Then there are at most $2k$ lines in each direction defined by these queens. There might be some queens $\mathcal{Q}' = \{Q_1, Q_2, \ldots, Q_{q'}\}$ that don't contribute to any line. Then there are at most $\left\lfloor \frac{4k-q'}{2} \right\rfloor$ lines in each direction.

Lower Bound of n

Proof



Define a new line through each of the queens of $\mathcal{Q}'$, distributing them evenly amongst the four directions. This gives us a total of $\left\lfloor \frac{4k-q'}{2} \right\rfloor + \left\lceil \frac{q'}{4} \right\rceil \leq 2k$ *in each direction.*

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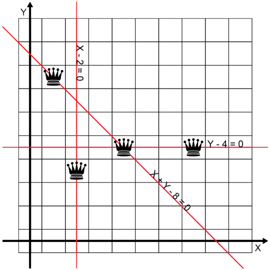
Proof



This gives us a set $\mathcal{L} = \{L_1, L_2, \ldots, L_{8k}\}$ of $8k$ lines. Let $l_i = 0$ be the equation defining L_i and consider the following polynomial:

$$f(x,y) = \prod_{i=1}^{8k} l_i$$

Note that $f(x,y) = 0$ *for all* x, y *since the queens must occupy or cover every square.*

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$$f(x, y) = \prod_{j=1}^{2k} (x - \alpha_j)(y - \beta_j)(x - y - \gamma_j)(x + y - \delta_j) \tag{1}$$

for suitable constants $\alpha_j, \beta_j, \gamma_j, \delta_j$.

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This contradiction completes the proof.

$\square$

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$$f(x,y)=\prod_{j=1}^{2k}(x-\alpha_j)(y-\beta_j)(x-y-\gamma_j)(x+y-\delta_j) \tag{1}$$

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This contradiction completes the proof.

$\square$

Proof extends to all other cases, that is, $m_3(n)\geq n$ for all n.

John W. Harris
541 Grove Lane
Santa Barbara, CA 93105

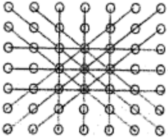
Dear Martin:

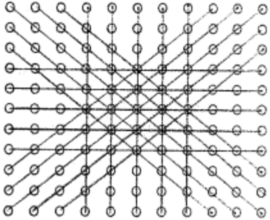
June 7, 1975

Thanks for your letter of May 30, with the excellent results on your new three in line problem. I think you have the minimums, and if $n+k$ counters are needed for the n^2 matrix, probably k tends to increase with n, but in a very irregular manner.

I can prove the minimum is at least n except when $n \equiv 3 \pmod 4$, in which case one less might be possible, but only if there are two counters or no counters in every row, column, or diagonal, a requirement so difficult to meet with only $n-1$ counters as to seem impossible, if every cell is to be crossed out.

Two counters can be said to cross out the orthogonal or diagonal that contains them, so consider first the simpler problem of crossing out every cell with n-1 lines in each direction. This can be done in a simple way when n ≡ 3(mod 4), as shown for 7 and 11:





For the remaining cases, first cross out K orthogonals each way, with any kind of spacing, regular or otherwise, and mark the cells that are left with x's, thus forming a rectangle of x's, perhaps something like this:

```
x x x   x x   x
x x x   x x   x

x x x   x x   x
x x x   x x   x
```

Consider only the perimeter x's:

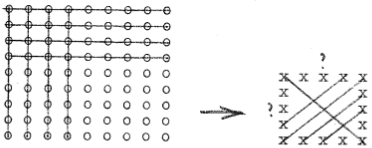
```
x x x   x x   x
x             x

x             x
x x x   x x   x
```

Case 1: n = 2m, K = m-1: The perimeter contains 4m of the x's, and the diagonal lines can't hit more than two each, for a total of 4m-4, so four cells are not on any of the lines, orthogonal or diagonal, and the corresponding counter problem has no solution.

This definition should be inserted before Case 1: K is the maximum number of pairs of counters one can form with less than n counters, so with $n = 2m$, we get $K = m-1$, and with $n = 4m+1$, we have $K = 2m$. Also, K is the number of lines in each direction.

Case 2. $n = 4m+1$, $K = 2m$. There are $8m$ of the perimeter x's this time, and the diagonals can cross out at most $8m$ of them, so each diagonal must account for exactly two x's, as we have already seen is possible when $n = 4m + 3$, $K = 2m + 1$. The number of diagonals touching the top x's is equal to the number of top x's, so there are $n - k = 2m + 1$ of them, and also $2m + 1$ diagonals touch the bottom. Since there are only $4m$ diagonals, two of them must reach from top to bottom. Similarly, two must reach from side to side, and our rectangle of x's is a square, with its two main diagonals connecting the corners.

Cross out the x's connected to the K lines of slope 1, and the same number of x's will remain at the left as there are at the top. If these are to be crossed out by the lines of slope -1, that will require an even number of such lines, not counting the main diagonal, and so this will only work when K is odd. QED, as they say. For example: $n = 9$, $K = 4$:



Best regards,

John

n	1	2	3	4	5	6	7	8	9
$m_3(n)$	1	4	4	4	6	6	8	**9**	**10**
n	10	11	12	13	14	15	16	17	18
$m_3(n)$	**10**	**12**	12	**[13,14]**	**[14,16]**	**[15,16]**	**[16,18]**	**[17,20]**	**[18,20]**

Table: $m_3(n)$, for small values of n. Brackets indicate lower and upper bounds.

n	1	2	3	4	5	6	7	8	9
$m_3(n)$	1	4	4	4	6	6	8	**9**	**10**
n	10	11	12	13	14	15	16	17	18
$m_3(n)$	**10**	**12**	12	**[13,14]**	**[14,16]**	**[15,16]**	**[16,18]**	**[17,20]**	**[18,20]**

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The C code that performed this brute-force search took around 900 3GHz-CPU hours to confirm that there is no good placement of 11 queens on board of side 11; our main result indicates that this was the smallest size we needed to test. We estimate that the corresponding search for a board of side 13 would require at least 70 thousand 3GHz-CPU hours.

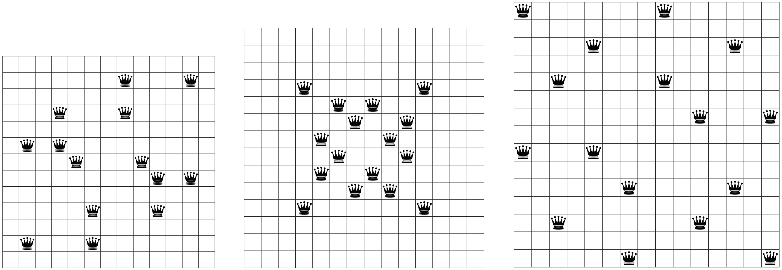


Figure: Maximal placements: 14 queens for $n = 13$; 16 queens for $n = 14$ and $n = 15$.

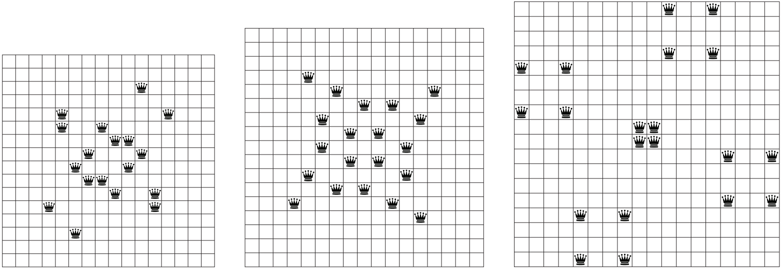
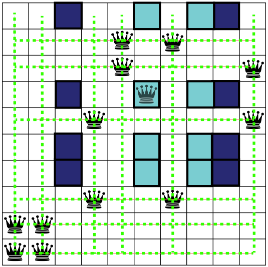
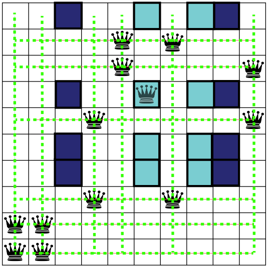


Figure: Maximal placements: 18 queens for $n = 16$; 20 queens for $n \in \{17, 18\}$.

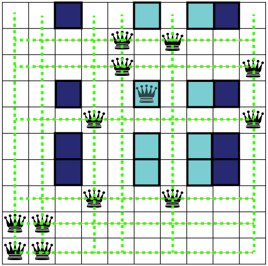
- Assume $n \geq 2$. Let $\mathcal{Q}$ be a good placement of size q.
- Set U to be the set of squares left uncovered by a line of slope 0 or ∞ and set $\mathcal{Q}'' \subseteq \mathcal{Q}$ to be those queens not involved in defining a line of slope 0 or ∞.



- Assume $n \geq 2$. Let $\mathcal{Q}$ be a good placement of size q.
- Set U to be the set of squares left uncovered by a line of slope 0 or ∞ and set $\mathcal{Q}'' \subseteq \mathcal{Q}$ to be those queens not involved in defining a line of slope 0 or ∞.
- Write $q'' = |\mathcal{Q}''|$. For any index $i \in [n]$ (respectively $j \in [n]$) let $C_i = \{(i,k) \in U : 1 \leq k \leq n\}$ (respectively $R_j = \{(k,j) \in U : 1 \leq k \leq n\}$).



- Let $a < b$ be the minimum and maximum indices, respectively, for which $C_i \neq \emptyset$. Set c to be the number of the C_i that are nonempty. Define $a' < b'$ and r analogously for the sets R_j.



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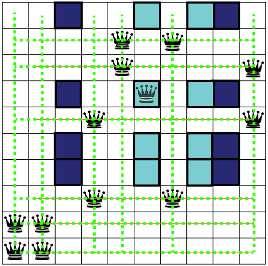
- Note that $c, r \geq n - \frac{q-q''}{2}$. In particular, $c \leq 1$ or $r \leq 1$ requires $q \geq 2(n-1)$. We therefore assume for the rest of the proof that $r, c \geq 2$.



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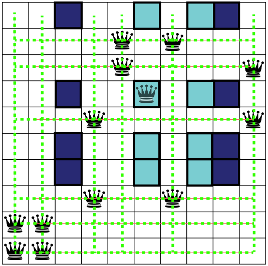
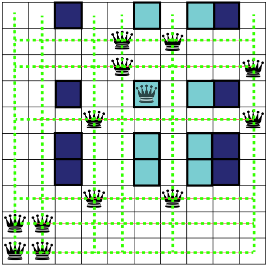
- Wlog, we may assume $b - a \geq b' - a'$ as otherwise we may rotate the placement by 90°.



- As $\mathcal{Q}$ is good, the squares of $C_a \cup C_b$ are either occupied or 'attacked' via a pair of queens that would define a line of slope ± 1. By definition, $|\mathcal{Q} \cap C_a| \leq 1$, $|\mathcal{Q} \cap C_b| \leq 1$; and so $|\mathcal{Q} \cap (C_a \cup C_b)| \leq \min\{q'', 2\}$.



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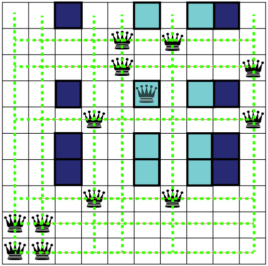
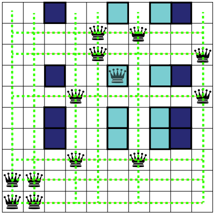
- There is at most one line of slope $+1$ that attacks two squares of $C_a \cup C_b$. Likewise for -1 slope. Each of the other lines of slope ± 1 defined by $\mathcal{Q}$ attack at most one square of $C_a \cup C_b$.



- $\mathcal{Q}$ defines at least $2r - 2 - \min\{q'', 2\}$ lines of slope ± 1. Furthermore,

$$\begin{aligned} 2r - 2 - \min\{q'', 2\} &\geq 2\left(n - \frac{q - q''}{2}\right) - 2 - q'' \\ &= 2n - q - 2. \end{aligned}$$



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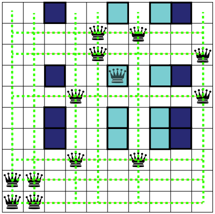

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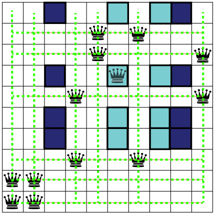

- q queens of $\mathcal{Q}$ define at most q lines of slope ± 1. Thus, $q \geq 2n - q - 2$, and so $q \geq n - 1$.
- Now restrict n to be even and reach a contradiction by assuming that $q \leq n - 1$.$\square$

n	1	2	3	4	5	6	7	8	9
$m_3(n)$	1	4	4	4	6	6	8	**9**	**10**
n	10	11	12	13	14	15	16	17	18
$m_3(n)$	**10**	**12**	12	**[13,14]**	**[14,16]**	**[14,16]**	**[16,18]**	**[17,20]**	**[18,20]**

Table: $m_3(n)$, for small values of n. Brackets indicate lower and upper bounds.

Data suggests that for n odd and $n \geq 3$ we might have $m_3(n) \geq n + 1$. Is it? If so, we've fallen short with both the algebraic and combinatorial proof.

Possible approach:

To get $m_3(n) \geq n+1$ for odd n: place n queens and be more careful in counting the lines defined by the set of queens placed, and then apply algebraic method.

Questions

- An upper bound for $m_3(n)$? Experimentally, we have $m_3(n) \leq 1.4108n + 5.87$.

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Thanks!