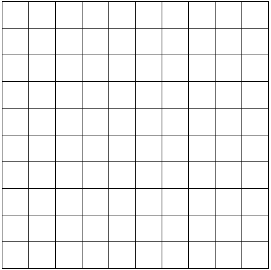
# Martin Gardner's Minimum No-Three-In-A-Line Problem

John Schmitt

Middlebury College, VT

joint work with Alec Cooper, Oleg Pikhurko, Greg Warrington

Consider a chessboard of size $n \times n$



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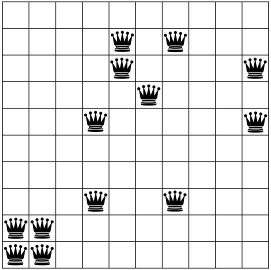

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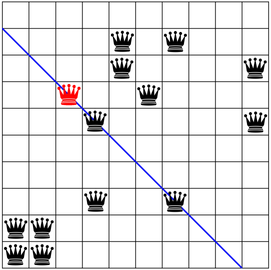

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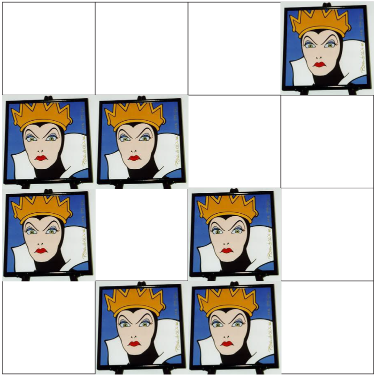

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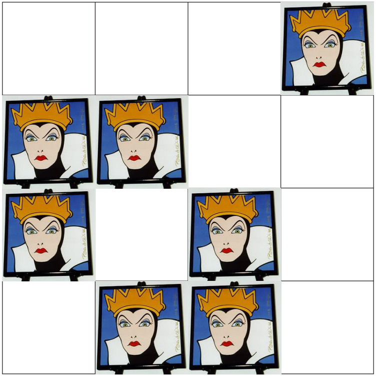

Question (Martin Gardner, 1976)

*How few queens ($m_3(n)$) can we use and still satisfy these properties?*

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*For all* $n \geq 1$, $m_3(n) > \frac{n}{2}$.

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## Proof.

Put $q$ queens on the board.
Each queen occupies one square and sees at most $4n - 4$ squares.
There are $n^2$ squares, each of which takes two queens to cover it or one queen to occupy it.

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$$\begin{aligned} \frac{1}{2}(4n-4)q + q &\geq n^2 \\ (2n-1)q &\geq n^2 \\ 2nq &> n^2 \\ q &> \frac{n}{2} \end{aligned}$$

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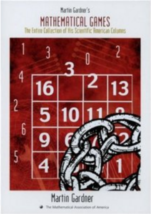
Can be improved since only a few queens see $4n - 4$ squares though each sees at least $3n - 3$ squares, but at best this gets to a bound of $\frac{2}{3}n$.

# Martin Gardner




- American mathematical journalist and author (October 21, 1914 – May 22, 2010)
- Wrote Mathematical Games column for *Scientific American* (1956-1981)
- Entire collection available on CD-ROM from the MAA for $55.95!

# A first-year seminar on Mathematical Games

- *The Combinatorics of Paper Folding*
  Activity: create stamp books, Dictators puzzle, Beelzebub puzzle, a tetraflexagon, and Sheep and Goats puzzle
  Ideas: permutations
- *A Matchbox Game-Learning Machine*
  Activity: build a machine for playing hexapawn
  Ideas: machine learning
- *The Binary System*
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- *Back from the Klondike and Other Problems*
  Activity: problems on the chessboard
  Ideas: combinatorics, algebra, algebraic geometry, linear programming, computer science . . .!!

Dear John: 2 June 75

Luckily you had sent me several pages, many years ago, about analysis of the heptominoes that tile and do not tile, so I've been able to summarize. You once said it was okay for me to use things mentioned in these early letters, so I've I called you procedure for checking the "Conway criterion," pictured the heptominoes to which it doesn't apply, identified the tilers, including the one you dubbed the "most interesting," and given as a problem for readers to finding of a tiling for one of the nontilers, combined with 3 x 3 squares. My other problem is finding a tiling for one of Penrose's poly- iamonds, that I call the loaded wheelbarrow (he gave me permission to use." It calls for 8 different orientations to form a fundamental region that tiles by translation.

I am postponing the column on "nonperiodic tiling" until next year, hoping that Penrose will give me permission by then to picture his two tiles. At that time I would very much wish to include your observations of this pattern -- and will do my best to alert you at least two months ahead of my deadline. So far, Penrose has no yet revealed to me the shapes of the tiles

Any chance of getting a look now at your two maps: the one for the Century puzzle, and for Dad's? I once did a column on sliding blocks, but one of these days I'll do another one, and I'd like to have these items on file with permission to use sometime. Does Knuth have them? I mention it because I think his next volume (on combinatoric algorithms) is going to include some sliding block material.

And OF COURSE I would like to get from you or somebody the new news about the angel problem!

I have no memory at this point of which of my column collections I've sent you, but there is a new paperback edition just out of the last one, The Sixth Book of etc., which contains my old column on sliding blocks. I have today put one in ship mail for you. Incidentally, I hope you don't mind that I have dedicated the seventh volume, Mathematical Carnival, to you, which is coming out this fall by Knopf. I have a few lines in the dedication, which I think you will like, but let me hold this off so there'll be some surprise left.

I enclose a piece about me in Time, occasioned by my April Fool hoax column, which produced about 2,000 letters from readers who didn't know it was a joke.

The current (June) column mentions Ulam's game of taking turns putting a counter on an n x n until one person wins by getting 3 in line, orthogonally or diagonally. This suggested to me the following problem, which I believe is new. What is the minimum number of counters that can be put on a square, no 3 in line, such that one more counter produces 3 in line?

I will enclose my best results through n = 12. I haven't been able to prove that my n = 8 (10 counters) is minimal, or to prove to at least n counters are required for all n, though I suspect it might not be hard to settle the latter conjecture one way or the other. It would be nice if the minimum for all odd n were N + 1, and for all even n, n or n + 2. If "line" is taken to any straight line on the field, the general problem becomes much more difficult. I note an article on the traditional "no-3-in-line" problem in the May J. of Comb. Theory, on which Guy has done some work. (I recently wrote him about all of this, asking for one of his papers on it, but haven't received a reply yet). He let me see a copy of his paper (forthcoming) on your Sylver coinage problem, about which I also have some comments from you in letters.

If you ever type up anything on Football, such as definitions of terms, etc., please keep me in mind for a copy. As I've said before, I'm holding off on this for a future column, for which SA would make a substantial payment (whether you like this or not), but anything you record about it I'd like to have on file, on a confidential basis. By fall, SA should know whether they've lost money of the six filmstrips I did (my work now completed, but strips not yet on market), which will determine how they will feel about trying a game.

I mistakenly told Penrose in a letter that the game agent here who handled Piet Hein, Tom Atwater, had retired from the game business. It is true that he quite his job with a lwa firm, to become dean of a business school, but I recently learned that he is continuing to act as agent for "select customers." Anyway, he can still be reached at his home, 15 Walden St., Concord, Mass 01742. In fact, I'll write him a letter today about Football, and ask him to contact you if he thinks it has market possibilities.

Best,

Dear Mr. Sands: 17 June 75

Many thanks for summarizing the results of that paper for me. I got lost on it at several spots.

Guy informs me that the reference to a solution for n = 12 is an error. On closer examination, the one solution that someone thought he found proved to have 3 in line! So -- no solutions, with 2n counters, are known beyond n = 10. It would be interesting, wouldn't it, if there are none higher than 1o?

I failed to keep a carbon of my letter, so I don't recall if I mentioned that I have been working on the simpler problem of the minimum number of counters, no 3 in line, such that one more counter puts 3 in line, but defining "line" as a row, column or diagonal.

My minimums, 3 through 12, are:

4,4,6,6,8,10,12,12,12.

I've been trying to prove that the minimum number of counters must be at least n, but the best I can do is prove it must be at least n - 1. (Actually, not my proof, but one from a correspondent, John Harris).

I can get 4-fold symmetry only for n = 4, 7 and 10,11, 12. The following pattern solves 10,11,12.

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### Lemma

*Let $\mathbb{F}$ be an arbitrary field, and let $f = f(x)$ be a polynomial in $\mathbb{F}[x]$. Suppose the degree of $f$ is $t$ (thus the $x^t$ coefficient of $f$ is nonzero). Then, if $S$ is a subset of $\mathbb{F}$ with $|S| > t$, there is an $s \in S$ so that*

$$f(s) \neq 0.$$

# Lower Bound of $n$

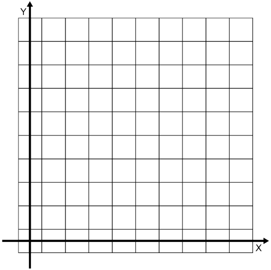
**Theorem (Cooper, Pikhurko, S. , Warrington, '14)**

*For any* $n \geq 1$*, we have* $m_3(n) \geq n$*, except when* $n \equiv 3 \pmod 4$ *when* $m_3(n) \geq n - 1$.

# Lower Bound of $n$

**Proof**

*We restrict our attention to* $n = 4k + 1$.




*Place our chessboard into the standard Cartesian coordinate system.*

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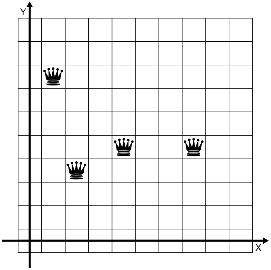
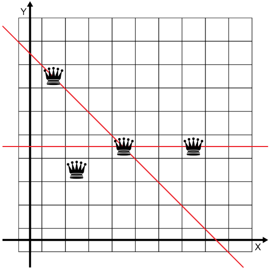



*Suppose a set $\mathcal{Q}$ of $q \leq 4k$ queens have been placed, and that these queens satisfy the requirements.*

# Lower Bound of $n$

## Proof




*Then there are at most $2k$ lines in each direction defined by these queens.*

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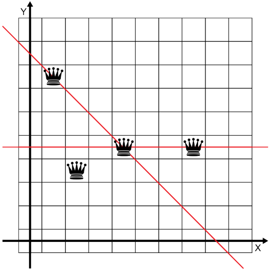
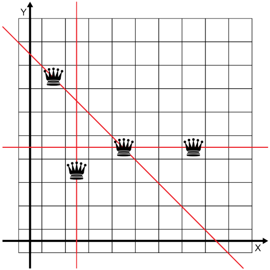



*Then there are at most $2k$ lines in each direction defined by these queens. There might be some queens $\mathcal{Q}' = \{Q_1, Q_2, \ldots, Q_{q'}\}$ that don't contribute to any line. Then there are at most $\left\lfloor \frac{4k-q'}{2} \right\rfloor$ lines in each direction.*

# Lower Bound of $n$

**Proof**




*Define a new line through each of the queens of $\mathcal{Q}'$, distributing them evenly amongst the four directions. This gives us a total of* $\left\lfloor \frac{4k-q'}{2} \right\rfloor + \left\lceil \frac{q'}{4} \right\rceil \leq 2k$ *in each direction.*

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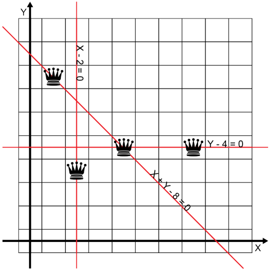


*This gives us a set* $\mathcal{L} = \{L_1, L_2, \ldots, L_{8k}\}$ *of* $8k$ *lines. Let* $\ell_i = 0$ *be the equation defining* $L_i$ *and consider the following polynomial:*

$$f(x, y) = \prod_{i=1}^{8k} \ell_i$$

Note that $f(x, y) = 0$ for all $(x, y)$, where $1 \leq x, y \leq 4k + 1$, since the queens must occupy or cover every square.

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$$f(x, y) = \prod_{j=1}^{2k} (x - \alpha_j)(y - \beta_j)(x - y - \gamma_j)(x + y - \delta_j) \tag{1}$$

for suitable constants $\alpha_j, \beta_j, \gamma_j, \delta_j$.

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In particular, the coefficient of the top-degree term $x^{4k}y^{4k}$ is $\pm\binom{2k}{k} \neq 0$.

**Theorem (Combinatorial Nullstellensatz Part 2, the Non-vanishing Corollary, N. Alon, 1999)**

*Let $\mathbb{F}$ be an arbitrary field, and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \ldots, x_n]$.*
*Suppose the degree $\deg(f)$ of $f$ is $\sum_{i=1}^{n} t_i$, where each $t_i$ is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in $f$ is nonzero.*
*Then, if $S_1, \ldots, S_n$ are subsets of $\mathbb{F}$ with $|S_i| > t_i$, there are $s_1 \in S_1, \ldots, s_n \in S_n$ so that*

$$f(s_1, \ldots, s_n) \neq 0.$$

so by the Combinatorial Nullstellensatz (Part II) with $t_1 = t_2 = 4k$ and $S_1 = S_2 = \{1, \ldots, 4k + 1\}$, we must have $s_1 \in S_1$ and $s_2 \in S_2$ such that $f(s_1, s_2) \neq 0$.

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This contradiction completes the proof.

$\square$

Proof extends to all other cases.

John W. Harris
541 Grove Lane
Santa Barbara, CA 93105

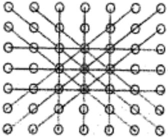
Dear Martin:

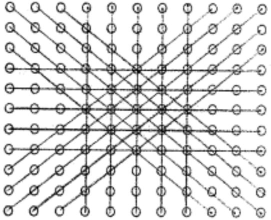
June 7, 1975

Thanks for your letter of May 30, with the excellent results on your new three in line problem. I think you have the minimums, and if n+k counters are needed for the $n^2$ matrix, probably k tends to increase with n, but in a very irregular manner.

I can prove the minimum is at least n except when $n \equiv 3 \pmod 4$, in which case one less might be possible, but only if there are two counters or no counters in every row, column, or diagonal, a requirement so difficult to meet with only n-1 counters as to seem impossible, if every cell is to be crossed out.

Two counters can be said to cross out the orthogonal or diagonal that contains them, so consider first the simpler problem of crossing out every cell with n-1 lines in each direction. This can be done in a simple way when $n \equiv 3 \pmod 4$, as shown for 7 and 11:





For the remaining cases, first cross out K orthogonals each way, with any kind of spacing, regular or otherwise, and mark the cells that are left with x's, thus forming a rectangle of x's, perhaps something like this:

```
x x x   x x   x
x x x   x x   x

x x x   x x   x
x x x   x x   x
```

Consider only the perimeter x's:

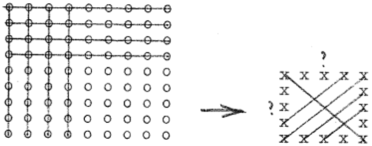
```
x x x   x x   x
x             x

x             x
x x x   x x   x
```

Case 1: $n = 2m$, $K = m-1$: The perimeter contains 4m of the x's, and the diagonal lines can't hit more than two each, for a total of 4m-4, so four cells are not on any of the lines, orthogonal or diagonal, and the corresponding counter problem has no solution.

This definition should be inserted before Case 1: K is the maximum number of pairs of counters one can form with less than n counters, so with $n = 2m$, we get $K = m-1$, and with $n = 4m+1$, we have $K = 2m$. Also, K is the number of lines in each direction.

Case 2. $n = 4m+1$, $K = 2m$. There are $8m$ of the perimeter x's this time, and the diagonals can cross out at most $8m$ of them, so each diagonal must account for exactly two x's, as we have already seen is possible when $n = 4m + 3$, $K = 2m + 1$. The number of diagonals touching the top x's is equal to the number of top x's, so there are $n - k = 2m + 1$ of them, and also $2m + 1$ diagonals touch the bottom. Since there are only $4m$ diagonals, two of them must reach from top to bottom. Similarly, two must reach from side to side, and our rectangle of x's is a square, with its two main diagonals connecting the corners.

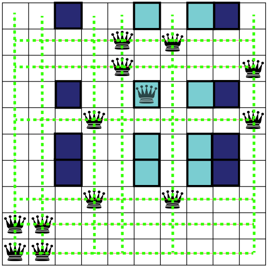
Cross out the x's connected to the K lines of slope 1, and the same number of x's will remain at the left as there are at the top. If these are to be crossed out by the lines of slope $-1$, that will require an even number of such lines, not counting the main diagonal, and so this will only work when K is odd. QED, as they say. For example: $n = 9$, $K = 4$:



Best regards,

John

- Assume $n \geq 2$. Let $\mathcal{Q}$ be a good placement of size $q$.
- Set $U$ to be the set of squares left uncovered by a line of slope 0 or $\infty$ and set $\mathcal{Q}'' \subseteq \mathcal{Q}$ to be those queens not involved in defining a line of slope 0 or $\infty$.



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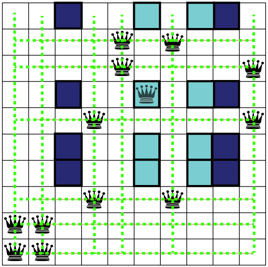
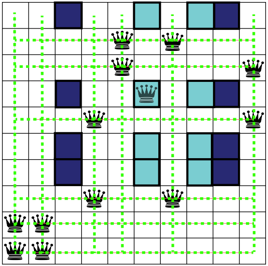
- Set $U$ to be the set of squares left uncovered by a line of slope 0 or $\infty$ and set $\mathcal{Q}'' \subseteq \mathcal{Q}$ to be those queens not involved in defining a line of slope 0 or $\infty$.
- Write $q'' = |\mathcal{Q}''|$. For any index $i \in [n]$ (respectively $j \in [n]$) let $C_i = \{(i,k) \in U : 1 \leq k \leq n\}$ (respectively $R_j = \{(k,j) \in U : 1 \leq k \leq n\}$).



- Let $a < b$ be the minimum and maximum indices, respectively, for which $C_i \neq \emptyset$. Set $c$ to be the number of the $C_i$ that are nonempty. Define $a' < b'$ and $r$ analogously for the sets $R_j$.



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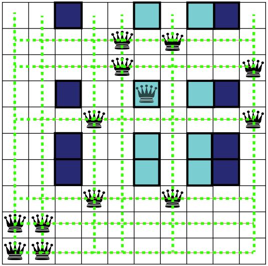
- Note that $c, r \geq n - \frac{q-q''}{2}$. In particular, $c \leq 1$ or $r \leq 1$ requires $q \geq 2(n-1)$. We therefore assume for the rest of the proof that $r, c \geq 2$.



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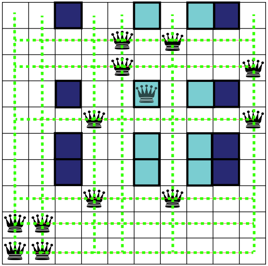
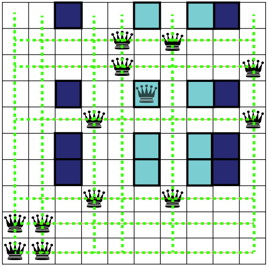
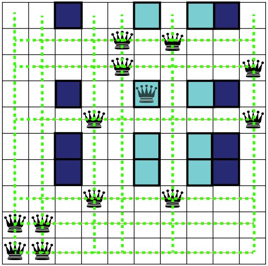
- Wlog, we may assume $b - a \geq b' - a'$ as otherwise we may rotate the placement by $90^\circ$.



- As $\mathcal{Q}$ is good, the squares of $C_a \cup C_b$ are either occupied or 'attacked' via a pair of queens that would define a line of slope $\pm 1$. By definition, $|\mathcal{Q} \cap C_a| \leq 1$, $|\mathcal{Q} \cap C_b| \leq 1$; and so $|\mathcal{Q} \cap (C_a \cup C_b)| \leq \min\{q'', 2\}$.

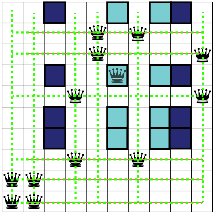


- As $\mathcal{Q}$ is good, the squares of $C_a \cup C_b$ are either occupied or 'attacked' via a pair of queens that would define a line of slope $\pm 1$. By definition, $|\mathcal{Q} \cap C_a| \leq 1$, $|\mathcal{Q} \cap C_b| \leq 1$; and so $|\mathcal{Q} \cap (C_a \cup C_b)| \leq \min\{q'', 2\}$.
- There is at most one line of slope $+1$ that attacks two squares of $C_a \cup C_b$. Likewise for $-1$ slope. Each of the other lines of slope $\pm 1$ defined by $\mathcal{Q}$ attack at most one square of $C_a \cup C_b$.



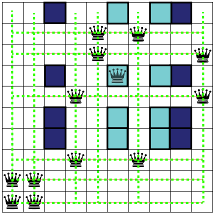
- $\mathcal{Q}$ defines at least $2r - 2 - \min\{q'', 2\}$ lines of slope $\pm 1$. Furthermore,

$$2r - 2 - \min\{q'', 2\} \geq 2\left(n - \frac{q - q''}{2}\right) - 2 - q''$$
$$= 2n - q - 2.$$



- $\mathcal{Q}$ defines at least $2r - 2 - \min\{q'', 2\}$ lines of slope $\pm 1$. Furthermore,

$$\begin{aligned} 2r - 2 - \min\{q'', 2\} &\geq 2\left(n - \frac{q - q''}{2}\right) - 2 - q'' \\ &= 2n - q - 2. \end{aligned}$$



- $q$ queens of $\mathcal{Q}$ define at most $q$ lines of slope $\pm 1$. Thus, $q \geq 2n - q - 2$, and so $q \geq n - 1$.

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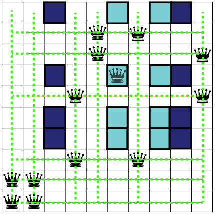

- $q$ queens of $\mathcal{Q}$ define at most $q$ lines of slope $\pm 1$. Thus, $q \geq 2n - q - 2$, and so $q \geq n - 1$.
- Now restrict $n$ to be even and reach a contradiction by assuming that $q \leq n - 1$.$\square$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $m_3(n)$ | 1 | 4 | 4 | 4 | 6 | 6 | 8 | **9** | **10** |
| $n$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| $m_3(n)$ | **10** | **12** | 12 | **[13,14]** | **[14,16]** | **[14,16]** | **[16,18]** | **[17,20]** | **[18,20]** |

Table: $m_3(n)$ for small values of $n$. Brackets indicate lower and upper bounds.

**Cooper, Pikhurko, S., Warrington - and brute-force search**

The C code that performed this brute-force search took around 900 3GHz-CPU hours to confirm that there is no good placement of 11 queens on board of side 11; our main result indicates that this was the smallest size we needed to test. We estimate that the corresponding search for a board of side 13 would require at least 70 thousand 3GHz-CPU hours.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $m_3(n)$ | 1 | 4 | 4 | 4 | 6 | 6 | 8 | **9** | **10** |
| $n$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| $m_3(n)$ | **10** | **12** | 12 | 14 | 15 | 16 | 17 | 18 | 18 |

Table: $m_3(n)$ for all values of $n$ where it is known precisely

**Cooper, Pikhurko, S., Warrington - and brute-force search**

Rob Pratt - and Integer Linear Programming

# An e-mail from Maggie McLoughlin

On 8/25/14, 8:58 PM, "Maggie McLoughlin"
<mam@CS.Stanford.EDU> wrote:
Dear Alec, Oleg, John, and Gregory!
Last Friday I came across your appealing paper about Gardner's problem of no-3-queens-in-a-line ... and it led to a very pleasant weekend indeed. As it happens, I'm currently writing a section of The Art of Computer Programming, Volume 4B, that deals with "satisfiability" and the revolutionary methods by which "SAT solvers" have been able to help resolve combinatorial problems of many kinds. So I realized that this queens problem makes an excellent test case for the algorithms I am discussing.

# An e-mail from Maggie McLoughlin

In particular, I can verify the lower bound for n=11, which you said took 900 CPU hours on a 3GHz machine: The SAT solver I gave it to was able to prove unsatisfiability, of the relevant clauses for placement of 11 queens, in two days or so. (My machine is only only 75% as fast as yours; but I can tell you precisely that the run did 8.9 teramems of work — namely, 8.9 trillion accesses to memory.) It explored 93 million nodes of an implicit search tree and learned 71 million clauses during the run.

....

Anyway, here are some of the solutions that I found:

....

# An e-mail from Maggie McLoughlin

These examples give new upper bounds of n+1 for n=14 thru n=20 ... and — big big surprise! — the EXACT value 18 for n=18! That was pure luck: I asked the machine only for a solution that has 19 or fewer, and it stumbled upon this one after five minutes. Then I asked it for a solution that has 18 or fewer, and it ran for several hours without success(!).
I soon noticed that (1) there usually are many, many solutions; and (2) lots of the solutions are especially nice because they put queens ONLY ON WHITE SQUARES! Therefore I made a special version, which obviously ran much faster, limiting the search only to such cases. Optimum solutions (or at least, solutions that achieve the best upper bound so far known) were quickly found in all cases except when n=12; no such placement of 12 queens on the white squares of a 12x12 is possible.

# An e-mail from Maggie McLoughlin

Looking further, I noticed that not only were queens only on white squares, but they appeared only in odd-numbered rows! So that reduced the search space to the fourth root of its original size. You will notice that every solution listed above has this property, except in the case of n=12. That 12x12 solution turned out to have eightfold symmetry, although I didn't specify any symmetry. Thus I wonder if it agrees with the solution(s) that you know, or if there are any unsymmetrical solutions for 12 queens on the 12x12. With extra work I could search this case exhaustively (but I don't have time — my budget for weekend recreation is already shot).

# An e-mail from Maggie McLoughlin

Some of the solutions above were discovered before I made this improvement to the program's upperbound heuristic. But then I could speed everything up tremendously.
Finally I realized that I could put even more contraints on solutions, but still get placements of n+1 queens on the odd boards n=21, 23, 25, etc., in a fraction of second. Using a different random seed, I'd get further solutions — too fast to bother writing them down.

# An e-mail from Maggie McLoughlin

Then I boiled the whole thing down to a simple problem on permutations of m elements, for boards of size 2m-1. That problem seems to have zillions of solutions ... I mean, zillions of ways to place 2m queens, as illustrated above for odd boards ... although I haven't been able to see a pattern that generalizes to arbitrary m. So I shall write a computer program to see how high I can go with this in a short amount of time. (The new program will use classical backtracking, not SAT solving.) I'll send you the code and the results when they are ready, if you're interested.

# An e-mail from Maggie McLoughlin

——————— Let me conclude this progress report with a request: I will almost surely be mentioning your work in my book, and I like to put FULL NAMES of all cited authors in the indexes to the books of this series, Therefore, I ask you to please tell me your full and complete name, including all middle names. (In Gregory's case, I'm pretty sure it is Gregory Saunders Warrington, based on the Harvard thesis. But John, you signed your Emory thesis only "John R. Schmitt"; what does the R stand for? And Alec, you still haven't written a thesis; what belongs between Alec and Cooper besides just S? And Oleg: For you I need to know also the spelling of your name in Cyrillic, INCLUDING the patronymic part. Readers often tell me that they appreciate this feature of my books, so I thank you in advance for any help you can give.

# An e-mail from Maggie McLoughlin

And of course I thank you also for the stimulation that your article provided, and the things it taught me about Alon's Nullstellensatz. Cordially, Don Knuth [Donald Ervin Knuth]

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $m_3(n)$ | 1 | 4 | 4 | 4 | 6 | 6 | 8 | **9** | **10** |
| $n$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| $m_3(n)$ | **10** | **12** | 12 | 14 | 15 | 16 | 17 | 18 | 18 |
| $n$ | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
| $m_3(n)$ | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 26 | 28 |

Table: $m_3(n)$ for all values of $n$ where it is known precisely

**Cooper, Pikhurko, S., Warrington - and brute-force search**

Rob Pratt - and Integer Linear Programming

Don Knuth - and SAT solver

See On-line Encyclopedia of Integer Sequences, Sequence A219760.

Question

*Data suggests that* $m_3(n) \geq n+1$ *when* $n$ *is odd. Is it?*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $m_3(n)$ | 1 | 4 | 4 | 4 | 6 | 6 | 8 | **9** | **10** |
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### Question

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| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $m_3(n)$ | 1 | 4 | 4 | 4 | 6 | 6 | 8 | **9** | **10** |
| $n$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| $m_3(n)$ | **10** | **12** | 12 | 14 | 15 | 16 | 17 | 18 | 18 |
| $n$ | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
| $m_3(n)$ | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 26 | 28 |

Table: $m_3(n)$ for all values of $n$ where it is known precisely

### Question

*Data suggests that* $m_3(n) \geq n + 1$ *when* $n$ *is odd. Is it?*

### Question

*Can you give an upper bound on* $m_3(n)$*?*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $m_3(n)$ | 1 | 4 | 4 | 4 | 6 | 6 | 8 | **9** | **10** |
| $n$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| $m_3(n)$ | **10** | **12** | 12 | 14 | 15 | 16 | 17 | 18 | 18 |
| $n$ | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
| $m_3(n)$ | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 26 | 28 |

Table: $m_3(n)$ for all values of $n$ where it is known precisely

### Question

*Data suggests that $m_3(n) \geq n + 1$ when $n$ is odd. Is it?*

### Question

*Can you give an upper bound on $m_3(n)$?*

### Question

*What is $m_3(28)$?*

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $m_3(n)$ | 1 | 4 | 4 | 4 | 6 | 6 | 8 | **9** | **10** |
| $n$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| $m_3(n)$ | **10** | **12** | 12 | 14 | 15 | 16 | 17 | 18 | 18 |
| $n$ | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
| $m_3(n)$ | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 26 | 28 |

Table: $m_3(n)$ for all values of $n$ where it is known precisely

## Question

*Might you prove that $m_3(34) = 34$ and that there is a solution of the type specified by Knuth here, A219760?*

THANKS!