Minimum Saturated Graphs & Ramsey Graphs

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joint work with
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Definition

Given a family of graphs $\mathcal{F}$, a graph G is **$\mathcal{F}$-saturated** if

for every $F \in \mathcal{F}, F \not\subset G$ and

for some $F \in \mathcal{F}, F \subset G + e$ for any $e \in E(\overline{G})$.

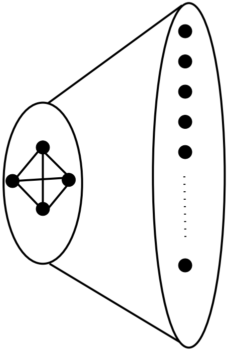
Problem

Determine the minimum number of edges of an n-vertex $\mathcal{F}$-saturated graph, denote this number by $sat(n, \mathcal{F})$.

Theorem (Erdős, Hajnal, Moon - '64)

$$sat(n, K_k) = (k-2)(n-1) - \binom{k-2}{2}, \quad n \geq k.$$

Furthermore, the only K_k-saturated graph with this many edges is $K_{k-2} + \overline{K}_{n-k+2}$.



Limiting executive compensation

Subsequently, Hajnal ('65) investigated K_k-saturated graphs without conical vertices.
Other results for K_k-saturated graphs with restrictions on maximum degree are given by: Hanson and Seyffarth ('84), Duffus and Hanson ('86), Erdős and Holzman ('94), Füredi and Seress ('94), and Alon, Erdős, Holzman and Krivelevich ('96).

Theorem (Barefoot et al. - '95)

A K_3-saturated graph which is not a star must have at least $2n - 5$ edges.

Difficulties and Hereditary Properties Lacking

Quote from Erdős, Hajnal and Moon:
"One of the difficulties of proving these conjectures may be that the obvious extremal graphs are certainly not unique, which fact may make an induction proof difficult."

- $sat(n, F) \not\leq sat(n+1, F)$
- $\mathcal{F}_1 \subset \mathcal{F}_2 \not\Rightarrow sat(n, \mathcal{F}_1) \geq sat(n, \mathcal{F}_2)$
- $F' \subset F \not\Rightarrow sat(n, F') \leq sat(n, F)$

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- $sat(2k-1, P_4) = k+1$ and $sat(2k, P_4) = k$
- $sat(n, \{P_5, S_4\}) = n-1 > sat(n, P_5)$
- $sat(n, K_4) = 2n-3$ but $sat(n, K_5 - S_3) \leq \frac{3}{2}n$

Best known upper bound

Theorem (Kászonyi and Tuza - '86)

Let $\mathcal{F}$ be a family of graphs. Set

$$u = u(\mathcal{F}) = \min\{|V(F)| - \alpha(F) - 1 : F \in \mathcal{F}\}$$

$$s = s(\mathcal{F}) = \min\{e(F') : F' \subseteq F \in \mathcal{F}, \alpha(F') = \alpha(F), |V(F')| = \alpha(F) + 1\}.$$

Then

$$sat(n, \mathcal{F}) \leq (u + \frac{s-1}{2})n - \frac{u(s+u)}{2}.$$

They considered a clique on u vertices joined to an $(s-1)$-regular graph.

$$sat(n, \mathcal{F}) = O(n)$$

Best known lower bound

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Best known lower bound

????

A trivial lower bound:

$$sat(n, F) \geq \frac{\delta(F)-1}{2} n$$

Best known lower bound

????

Problem

For an arbitrary graph F determine a non-trivial lower bound on $sat(n, F)$.

Definitions

$F \rightarrow (F_1, \ldots, F_t)$ if any t coloring of $E(F)$ contains a monochromatic F_i-subgraph of color i for some $i \in [t]$.

F is $(F_1, \ldots, F_t)$-*Ramsey-minimal* if $F \rightarrow (F_1, \ldots, F_t)$ but for any proper subgraph F' of F, $F' \not\rightarrow (F_1, \ldots, F_t)$.

Let $\mathcal{R}_{min}(F_1, \ldots, F_t) = \{F : F \text{ is } (F_1, \ldots, F_t) - Ramsey - minimal\}$.

Main problem

Conjecture (Hanson and Toft, '87)

Given $t \geq 2$ *and numbers* $m_i \geq 3, i \in [t]$.
Let $r = r(K_{m_1}, \ldots, K_{m_t})$ *be the classical Ramsey number. Then*

$$sat(n, \mathcal{R}_{min}(K_{m_1}, \ldots, K_{m_t})) = (r-2)(n-1) - \binom{r-2}{2}.$$

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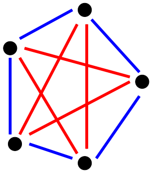
For $t = 1$ or $m_2 = m_3 = \ldots m_t = 2$, the conjecture reduces to the theorem of Erdős, Hajnal, and Moon.

Upper bound (example for $sat(n, \mathcal{R}_{min}(K_3, K_3))$:



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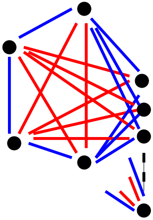

'Clone' a vertex.

Lower bound for $sat(n, \mathcal{R}_{min}(K_k, K_k))$ follows from:

Theorem (Burr, Erdős, Lovász - '76; Fox, Lin - '06)

The minimum degree of a graph in $\mathcal{R}_{min}(K_k, K_k)$ is at least $(k-1)^2$.

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$$sat(n, \mathcal{R}_{min}(K_k, K_k)) \geq \frac{(k-1)^2 - 1}{2} n.$$

This is miserable compared to upper bound.

$\mathcal{R}_{min}(K_3, K_3)$-saturated graphs

$\mathcal{R}_{min}(K_3, K_3)$-saturated graphs were investigated by Galluccio, Simonovits, and Simonyi ('95) (using slightly different terminology) and Szabó ('96). They gave various product constructions for such graphs. These constructions generally produce graphs with 'many' edges.

Theorem (GSS-'95)

If G_1 and G_2 are two non-bipartite K_3-saturated graphs, then $G_1 + G_2$ is a $\mathcal{R}_{min}(K_3, K_3)$-saturated graph.

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Theorem (GSS-'95)

A $\mathcal{R}_{min}(K_3, K_3)$-saturated graph has minimum degree at least 4.

This gives a slight improvement to the previous trivial lower bound for the case $sat(n, \mathcal{R}_{min}(K_3, K_3))$.

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PROOF: Apply Turán's theorem.

Theorem (Chen, Ferrara, Gould, Magnant, S.)

For $n \geq 56$, $sat(n, \mathcal{R}_{min}(K_3, K_3)) = 4n - 10$.

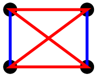
This confirms the first non-trivial case of Hanson-Toft Conjecture.

Theorem (Chen, Ferrara, Gould, Magnant, S.)

For $n \geq 11$, $sat(n, \mathcal{R}_{min}(K_3, P_3)) = \lfloor \frac{5n}{2} \rfloor - 5$.

A natural upper bound for $sat(n, \mathcal{R}_{min}(K_3, P_3))$

For a natural upper bound, one might think of Chvátal's 'clique vs. tree' Theorem for the case $r(K_3, P_3)$.



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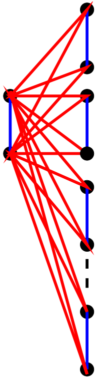
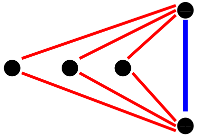
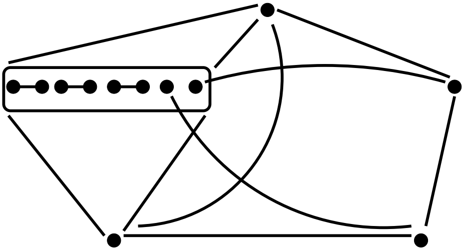


A fact for $\mathcal{R}_{min}(K_3, P_3)$-saturated graphs

Fact: In any good red/blue-coloring of a $\mathcal{R}_{min}(K_3, P_3)$-saturated graph any edge lying in three or more triangles must be colored blue (and so the other edges in the triangles must be colored red).



A better (and a best) upper bound for $sat(n, \mathcal{R}_{min}(K_3, P_3))$



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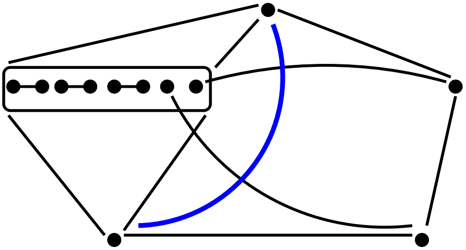

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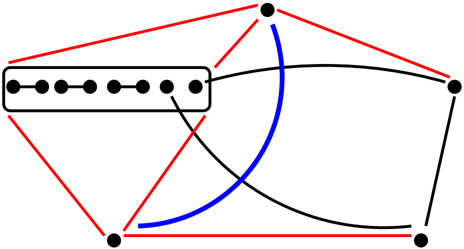

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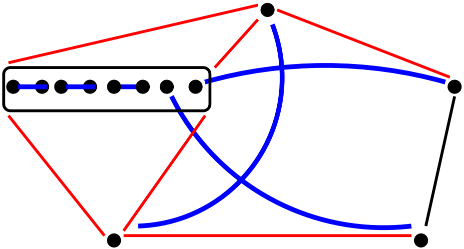

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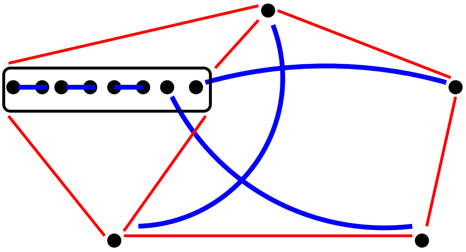

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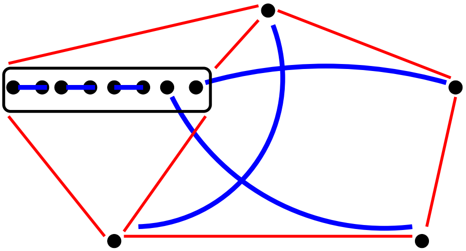

So the coloring is unique, and the graph is $\mathcal{R}_{min}(K_3, P_3)$-saturated. This provides the upper bound

A proof of the lower bound

Let G be an n-vertex $\mathcal{R}_{min}(K_3, P_3)$-saturated graph with minimum number of edges. Consider a good coloring of G with maximum number of red edges. Let G_b denote the blue graph and G_r the red graph.

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Claim: G_r is 2-connected.

A proof of the lower bound

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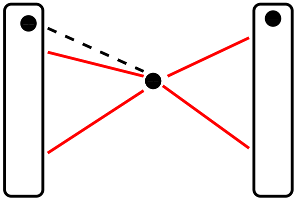
- Obviously G_r is connected as otherwise G_b would contain a complete bipartite graph - a contradiction.
- Suppose G_r has connectivity 1.

Claim: G_r is 2-connected.



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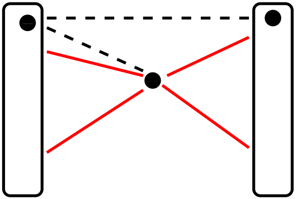

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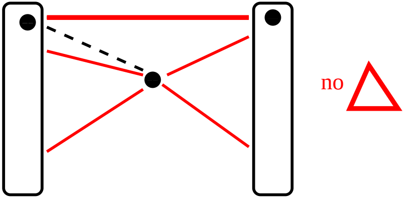

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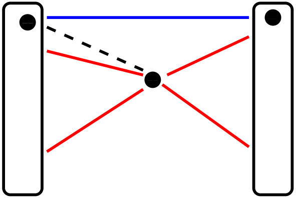

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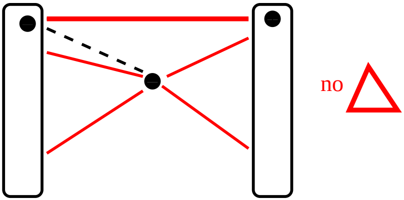
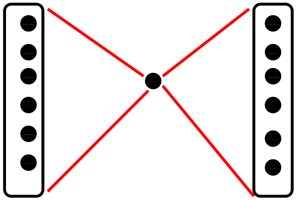


and contradicts choice.

Claim: G_r is 2-connected.



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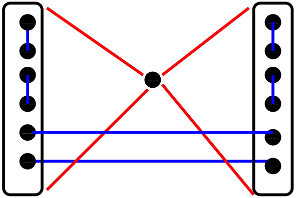

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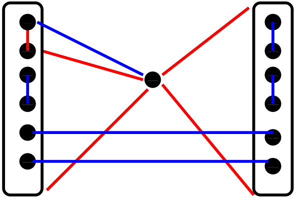

Color Swap

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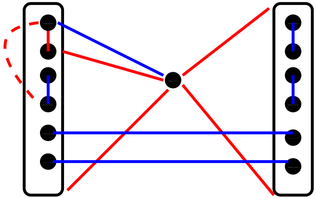

We see that G is not $\mathcal{R}_{min}(K_3, P_3)$-saturated. Establishes claim that G_r is 2-connected.

Completing the proof

Case: n is odd

G_r is K_3-saturated — if not, either add a red edge to G or re-color blue edge of G red. Apply a theorem of Barefoot et al. ('95) to G_r to obtain that G_r has at least $2n-5$ edges. Also, G_b must have $\lfloor \frac{n}{2} \rfloor$ edges. Thus, G has at least $\lfloor \frac{5n}{2} \rfloor - 5$ edges.

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Case: n is even
We omit here.
□

A few words about the proof of $sat(n, \mathcal{R}_{min}(K_3, K_3)) = 4n - 10$.

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A few words about the conjecture.

Open problems and questions:

- Hanson-Toft conjecture remains open in general.
- Does every $\mathcal{R}_{min}(K_3, K_3)$-saturated graph contain a K_4? [GSS-'95]
- If G is a $\mathcal{R}_{min}(K_3, K_3)$-saturated graph containing a K_5 must it also contain a $K_6 - e$? [GSS-'95]
- Can one find a finite set $Q_1, Q_2, \ldots, Q_m$ of $\mathcal{R}_{min}(K_3, K_3)$-saturated graphs so that every $\mathcal{R}_{min}(K_3, K_3)$-saturated graph contains at least one of them? [GSS-'95]

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