RESEARCH STATEMENT
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1. Introduction

My research interests lie within combinatorics, and most often (although not always) within graph theory. The active area of extremal combinatorics and graph theory, where my main interests lie, has its beginnings in 1941 in a seminal paper [28] of Paul Turán.

To summarize the majority of my research in a colloquial fashion, I seek to construct graphs with desired properties that are, in some way, "efficient." Once a graph candidate is in hand, I seek to prove that there is no better candidate.

More formally, in extremal graph theory we seek to determine the relationship between various graph invariants, such as order, size, connectivity, minimum degree, maximum degree, chromatic number and diameter. We also wish to determine the maximum, or minimum, value of a particular invariant that ensures that a graph has a certain property. Thus, "given a property P and an invariant f for a class H of graphs, determine the least value m for which every graph G in H with $f(G) > m$ has property P" ([3]). An *extremal graph* is a graph without property P and with $f(G) = m$.

2. Results

To better place my results in their proper context, we give a short history of extremal graph theory.

Turán determined the maximum number of edges a graph G can contain so that G does not contain a clique of order t, denoted K_t. Such a graph G is said to be *K_t-free.* If G is K_t-free and maximal (with respect to edges) then we may call it *K_t-saturated.* Turán determined this number, $ex(n, K_t)$, and provided the unique extremal graph, $T_{(n,t-1)}$.

Theorem 2.1. [28] *Turán's Theorem*
All graphs on n vertices with more edges than that contained in $T_{(n,t-1)}$ must contain a copy of K_t, and the unique extremal graph is $T_{(n,t-1)}$.

This theorem began the branch of extremal graph theory, an area that has proved useful to computer science, optimization theory and number theory. It was followed several years later by a cornerstone result of P. Erdős and A. Stone, which determined the value of $ex(n, K_{t(s)})$, where $K_{t(s)}$ denotes the complete t-partite graph with partite sets of size s. Later, when the concept of the chromatic number of a graph F was introduced, this result was extended [10], and is frequently referred to as the *The Fundamental Theorem of Extremal Graph Theory.*

Theorem 2.2. [9, 10] *Fundamental Theorem of Extremal Graph Theory*
Given a graph F with chromatic number $\chi(F) \geq 3$ and fixed ϵ, every graph on n vertices with $(1 - \frac{1}{\chi(F)-1} + \epsilon)\frac{n^2}{2}$ edges contains a copy of F.

While this theorem is very broad, it fails to address the situation when $\chi(F) = 2$. And while partial results exist, the problem of determining $ex(n, K_{s,t})$ (where $K_{s,t}$ denotes the complete bipartite graph with partite sets of sizes s and t) is completely open. It was Zarankiewicz that first addressed this problem in terms of $0-1$ matrices. He posed the equivalent problem of determining the maximum number of $1'$s in $m \times n$ $0-1$ matrix which has no $s \times t$ submatrix of all 1's. For an interesting account of this problem see [6].

Problem 1. *The problem of Zarankiewicz*
Determine the value of $ex(n, K_{s,t})$.

The two main problems which I have worked on may be considered as variations of the problem of Turán. Both of these problems were posed by Erdős, and others, in two seperate papers. These two problems, though posed about 27 years apart, have a close connection which was unknown prior to my work. I will give this connection after discussing each in detail. The theorems and problem above, along with other results in the area, have provided a blueprint for my investigation of these variations. A third and more recent problem is also discussed.

2.1. **The Turán Problem for Degree Sequences.** In joint work with M. Ferrara and R. Gould, we consider the following extremal problem as introduced by Erdős *et al.* in [8]. Let $\pi = (d_1, d_2, \ldots, d_n)$ be an n-element graphic sequence (that is there exists a graph with degree sequence π), and we'll call $\Sigma_{i=1}^{n} d_i$ the degree sum of π. Let F be a graph. Erdős *et al.* [8] sought to determine the smallest m such that any n-term graphic sequence π having degree sum at least m has a realization containing F as a subgraph. We denote this value m by $\sigma(F, n)$. Erdős, Jacobson and Lehel were interested when $F = K_t$. They conjectured that a graphic degree sequence π with degree sum at least $(t-2)(2n-t+1)+2$ has a realization containing a K_t. The conjecture arises from considering $E_{(n,t)} := K_{t-2} + \overline{K}_{n-t+2}$. The cases $t = 3, 4$ and 5 were proved separately (see respectively [8], [16] and [20], and [21]), and Li, Song and Luo [22] proved the conjecture true via linear algebraic techniques for $t \geq 6$. This collection of papers resolved the conjecture. However, we have been able to give a single, graph-theoretic proof of this result (see [11]).

Theorem 2.3. *For n sufficiently large, all n-term graphic degree sequences with degree sum at least $(t-2)(2n-t+1)+2$ has a realization containing a copy of K_t, and the unique extremal graph is $E_{(n,t)}$.*

Our proof of the Erdős, Jacobson, Lehel conjecture is fairly simple. Out of all realizations of π that meet the conditions, by induction we choose one with a large clique number. We then show that through a sequence of edge exchanges (where we replace two parallel edges by two parallel non-edges which occur on the same set of four vertices) we can obtain a realization with a larger clique number.

With the clique case solved we turned our attention to determinining the value of $\sigma(K_{t(s)}, n)$ - which I achieved for $s = 1, 2$ in my thesis [27] (and in [11]) and for $s \geq 3$ in [5] using a degree majorization approach.

In light of the Fundamental Theorem of Extremal Graph Theory, we are interested in determining $\sigma(F, n)$ for a given graph F. Towards this goal, we have a conjecture which relates the value of the function to the independence number of F (which gives rise to a function $u(F)$) and the structure of certain subgraphs of F (which gives rise to a function $d(F)$). We have been able to verify our conjecture for a broad class of graphs, those with independence number two [14], [15].

Conjecture 1. *Given a subgraph F and a fixed $\epsilon > 0$ every degree sequence of length $n \geq n_0 = n_0(\epsilon, F)$ with degree sum at least* $\max\{(n(2u(F^*) + d(F^*) - 1 + \epsilon) \mid F^* \subseteq F\}$, *contains a realization with a copy of F.*

We have also determined $\sigma(F, n)$ when F is the graph consisting of k triangles intersecting in a common point, see [12]. In joint work with J. Yin and G. Chen, we considered when F is the graph consisting of k t-cliques intersecting in a common r set [30].

In joint work with M. Ferrara, M. Jacobson and M. Siggers, we have completely solved the Zarankiewicz version for degree sequences. The problem is as follows, given a bigraphic sequence S, and a fixed bipartite graph F, we define $\sigma(F, m, n)$ to be the minimum integer k such that every bigraphic sequence $S = (A; B)$ with $|A| = m, |B| = n$ and degree sum at least k is has a realization containing F as a subgraph. We have determined $\sigma(K_{s,t}, m, n)$, $\sigma(P_t, m, n)$ and $\sigma(C_{2t}, m, n)$, see [13].

2.2. **Saturated Graphs.** A graph G is said to be *F-saturated* if G contains no copy of F and for every edge e in the complement of G, the graph $G + e$ contains a copy of F. Erdős, Hajnal and Moon [7] determined the unique K_t-saturated graph with the minimum number of edges. Determining the exact value of $sat(n, F)$, the minimum number of edges in an F-saturated graph, has been very difficult and is known for relatively few graphs. This is due to the fact that the function does not have the monotone properties which $ex(n, F)$ does. Thirty-five years ago Ollmann determined $sat(n, K_{2,2})$ in [24], and later Tuza [29] gave a shorter proof. In joint work with R. Gould, we adopt some of the techniques developed by Tuza and have determined the minimum number of edges in a $K_{t(2)}$-saturated graph of minimal minimum degree δ, denoted $sat(n, K_{t(2)}, \delta)$, (see [18]). It seems reasonable to assume that $sat(n, F, \delta)$ is an increasing function of δ. In joint work with O. Pikhurko, we have determined the asymptotic behavior of $sat(n, K_{2,3})$ (see [25]) - the first new result for complete bipartite graphs in many years.

In joint work with R. Gould and T. Łuczak ([17]), we have improved the upper bound previously determined in [1] for $sat(n, C_l)$, where C_l denotes a cycle of length l. (This was originally given as a problem by B. Bollobás in [3].) The upper bound follows via a construction, and we conjecture that our construction is asymptotically optimal for sufficiently large n and certain values of l.

2.3. **Relating the two problems.** In my thesis [27], I was the first to show that there exists a connection between the two functions, $sat(n, F)$ and $\sigma(F, n)$. Using an upper bound for the function $sat(n, F)$ as given by Kászonyi and Tuza [19] and the lower bound that follows from the construction given prior to Conjecture 1 for $\sigma(F, n)$, I posed the following.

Conjecture 2. *Let F be a graph and let n be a sufficiently large integer. Then*

$$2sat(n, F) < \sigma(F, n).$$

I have shown that the conjecture holds for graphs which satisfy a certain inequality.

2.4. **The transmission of consumable resources - graph pebbling.** Graph pebbling is a mathematical model for the transmission of consumable resources.

We start with a distribution of pebbles on the vertices of G, which may be thought of as an assignment of an integer amount of some specific consumable resource (such as petroleum, electricity, or water). We say that a *pebbling move* consists of removing two pebbles from a vertex and then placing one pebble on an adjacent vertex. That is, to move one unit of petroleum from one location to an adjacent location we use one unit of petroleum. Given a target vertex r, we say that r can be *reached* if after a sequence of pebbling moves it is possible to place a pebble on r. The *pebbling number* of $G, \pi(G)$, is the least integer m such that, regardless of how m pebbles are distributed on the vertices of G, after a sequence of pebbling moves we can reach any vertex. It is easy to see that $\pi(G) > n-1$ since placing each of $n-1$ pebbles on a distinct vertex leaves one vertex, r, without a pebble and no pebbling moves possible. Graphs for which $\pi(G) = n$ are known as Class 0 graphs and this class is the object of our consideration.

In work with undergraduate student Anna Blasiak, we have given a sufficient degree sum condition for a graph to be Class 0 [2]. A corollary to our result is that every graph with at least $\binom{n-1}{2}+2$ edges is Class 0. In on-going work, we are investigating the minimum size of Class 0 graphs. As a first step, as the pebbling number of a graph is related to the diameter of the graph, we have investigated the minimum number of edges in a graph with a fixed diameter and fixed minimum degree. In work in preparation with P. Dankelmann and L. Volkmann, we have been able to determine this number. In a specific case our result confirms that a construction given by Erdős [4] is indeed best possible and the corresponding conjecture given by Bondy and Murty.

3. Future Work

3.1. **Degree sequences.**

- Prove Conjecture 1, and the stronger version of this conjecture as given in [14], [15].
- Determine the value of $\sigma(F, n)$ when F is an r-uniform hypergraph. We have been able to obtain a lower bound, and our edge-exchange technique seems plausible in the $r = 3$ case as indicated by recent results of Kocay and Li.
- Prove Conjecture 2.

3.2. **Saturated graphs.**

- Prove that $sat(n, F, \delta)$ is a monotonically increasing function in δ.
- Prove that the construction for C_l-saturated graphs given in [17] is asymptotically optimal for large n and certain l.
- Determine the minimum number of edges that a subgraph of Q_n (the n-dimensional hypercube) must contain to be Q_m-saturated, where $m < n$. With an undergraduate student, Anthony Santolupo, I have investigated this problem, yet most cases remain open.

3.3. **Graph pebbling.**

- Prove that the minimum number of edges in a Class 0 graph goes to $\frac{3n}{2}$ as $n \to \infty$.

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