1 Introduction

My research interests lie within the area of Extremal Graph Theory, a sub-branch of Combinatorics. This active area of mathematics has its beginnings in 1941 in a seminal paper [27] of Paul Turán. Turán along with the celebrated Paul Erdős are considered the fathers of this area of mathematics and the active “Hungarian school” which has shared it with the world.

In extremal graph theory we seek to determine the relationship between various graph invariants, such as order, size, connectivity, minimum (maximum) degree, chromatic number and diameter. We also wish to determine the maximum, or minimum, value of a particular invariant that ensures that a graph has a certain property. Thus, “given a property $P$ and an invariant $f$ for a class $\mathcal{H}$ of graphs, determine the least value $m$ for which every graph $G$ in $\mathcal{H}$ with $f(G) > m$ has property $P$” ([2]).

Turán’s original question may now be stated. Given a graph $G = (V, E)$ where $V$ denotes a set of $n$ points called vertices, and $E$ a set of two element subsets of $V$ called edges, determine the maximum number of edges a graph $G$ can contain so that $G$ does not contain a clique (a subgraph with all possible edges) of order $t$, where such a clique is denoted $K^t$. Turán determined this number, $ex(n, K^t)$, and provided the unique extremal graph for this function. An extremal graph is a graph without property $P$ and with $f(G) = m$.

2 Current Research

The two main problems which I have worked on may be considered as Turán-type problems. Both of these problems were posed by Erdős, and others, in two separate papers. These two problems, though posed about 27 years apart, have a close connection which I will describe. For the first of these problems we have provided a complete solution, and the second made significant progress.

2.1 The Turán Problem for Degree Sequences

In joint work with M. Ferrara and R. Gould, we consider a variation of the classical Turán-type extremal problem as introduced by Erdős et al. in [9]. Let $\pi = (d_1, d_2, \ldots, d_n)$ be an $n$-element
graphical sequence, and $\sigma(\pi) = \Sigma_{i=1}^n d_i$, i.e. the degree sum. Let $F$ be a graph. Erdős et al. [9] sought to determine the smallest $m$ such that any $n$-term graphical sequence $\pi$ having $\sigma(\pi) \geq m$ has a realization containing $F$ as a subgraph (our desired property $P$). Such a sequence would be called potentially $F$-graphic. We denote this value $m$ by $\sigma(F, n)$. Erdős, Jacobson & Lehel were interested in $F = K_t^t$. They proposed the following split graph as the extremal graph: $G^* = K_t-t^t + \overline{K}^{n-t+2}$ (where $\overline{F}$ denotes the complement of the graph $F$). This graph is the unique realization of the degree sequence $\pi_1 = ((n-1)^t-2, (t-2)^{n-t+2})$ with degree sum equal to $m_1 = (t-2)(2n-t+1)$, and contains no copy of $K_t^t$. (The notation $(n-1)^t-2$ means $n-1$ repeated $t-2$ times in the degree sequence.) They thus conjectured, a graphical degree sequence $\pi$ with degree sum greater than $m_1$ has a realization containing a $K_t^t$. The cases $t = 3, 4$ and $5$ were proved separately (see respectively [9], [15] and [19], [20]), and Li, Song and Luo [21] proved the conjecture true via linear algebraic techniques for $t \geq 6$ and $n \geq \frac{t}{2} + 3$. This collection of papers resolved the conjecture. However, we have been able to give a single, graph-theoretic proof of this result (see [10]).

Our proof of the Erdős, Jacobson, Lehel conjecture hinges upon the following. First, the degree sum given guarantees the existence of $t$ vertices of degree $t-1$ - certainly necessary in creating a clique of order $t$. Second, we use a theorem from [15] which states that if $\pi$ has a realization containing a subgraph $F$, then there is a realization of $\pi$ containing $F$ on the vertices of highest degree. We are then able to use induction, and a process we call “edge swapping” (or 2-switching.) That is, we show that from a given realization of $\pi$ as a starting point we can repeatedly swap two parallel edges for two parallel non-edges that involve the same four vertices and arrive at a realization of $\pi$ containing a $t$-clique.

With the clique case solved we turned our attention to determining the value of $\sigma(K^t_s, n)$, where $K^t_s$ denotes the complete multi-partite graph with $t$ partite sets of size $s$. Let’s think of the extremal graph $G^*$ in the following manner. Let $H$ be a clique on $t-2$ vertices. Let $H'$ be a graph on $n - t + 2$ vertices having the maximal number of edges such that the degree sequence of $H'$ is not potentially $K_{1,1}$-graphic. The graph $G^*$ is then the join of $H$ and $H'$. With this point of view we propose more generally that if $H$ is a clique on $s(t-2)$ vertices, and $H'$ a graph on $n-st + 2s$ vertices having the maximal number of edges such that the degree sequence of $H'$ is not potentially $K_{s,s}$-graphic then $G^*$ is the extremal graph. This conjecture relies on knowing the values for $\sigma(K^t_s, n)$ as determined in [15] and [22]. We have confirmed this proposal in the $s = 2$ case, see [10], and more recently with G. Chen have been able to resolve all remaining cases, [5].

Quite recently, we have also been able to determine $\sigma(F_k, n)$ when $F_k$ is the friendship graph, that is the graph consisting of $k$ triangles intersecting in a common point, see [11]. We are also seeking to determine the value of this function for $k$ $t$-cliques intersecting in a common $r$ set.

For an arbitrary graph $F$ determining the value of $\sigma(F, n)$ is rather interesting. With M. Ferrara, we have been able to give a lower bound that relates $\sigma(F, n)$ to the independence number of $F$ and its degree sequence, see [13, 14]. The lower bound we give matches the value of the function for all computed values thus far. Moreover, we have determined the value of the function for all graphs with independence number two. It remains open for graphs with independence number larger than two. If we can prove that this lower bound is indeed sufficient then we will have an Erdős-Stone-Simonovits type theorem for this problem.

We have also done research into the Zarankawiecz version for degree sequences. That is given a bigraphic sequence $S$, and a fixed bipartite graph $H$, we say that $S$ is potentially $H$-bigraphic if there is some realization of $S$ containing $H$ as a subgraph. We define $\sigma(H, m, n)$ to be the minimum
integer $k$ such that every bigraphic sequence $S = (A; B)$ with $|A| = m$, $|B| = n$ and $\sigma(S) \geq k$ is potentially $H$-bigraphic. We have determined $\sigma(K_{st}, m, n)$, $\sigma(P_t, m, n)$ and $\sigma(C_{2t}, m, n)$, see [12].

2.2 Saturated Graphs

A graph $G$ is said to be $F$-saturated if $G$ contains no copy of $F$ and for every edge $e$ in the complement of $G$, the graph $G + e$ contains a copy of $F$. Thus Turán’s graph is the unique $K^t$-saturated graph with the maximum number of edges. About twenty years later Erdős, Hajnal and Moon [7] determined the unique $K^t$-saturated graph with the minimum number of edges, where this minimum is denoted $sat(n, K^t)$, to be $G^*$ - the same graph as the extremal graph for $\sigma(K^t, n)$! Thus we propose the same construction for determining $sat(n, K^t_s)$. That is, a clique on $s(t - 2)$ vertices joined to a graph on $n - st + 2s$ vertices which is minimally $K_{s,s}$-saturated. (There are a few small changes to this graph in fact that can be done to improve the edge count, which won’t be described here.) Unfortunately, determining the exact value of $sat(n, F)$ has been very difficult and is known for relatively few graphs. (For general bounds for graphs and hypergraphs see [28], [18], [24].) For $s = 2$ Ollmann provided the answer in [23], and later Tuza [29] gave a shorter proof. Adopting some of the techniques developed by Tuza we have been able to determine the minimum number of edges in a $K^t_s$-saturated graph of minimal minimum degree $\delta$, denoted $sat(n, K^t_s, \delta)$ , (see [17]). It seems reasonable to assume that $sat(n, K^t_s, \delta)$ is an increasing function of $\delta$, yet we have been unable to prove this so far. For $s \geq 3$ we have sought to determine the value for $sat(n, K_{s,s})$ which would enable us to solve the larger problem. With a specific construction which generalizes the graphs given by Ollmann we have been able to improve the upper bound given in [18] for $sat(n, K_{s,s})$, and thus in turn $sat(n, K^t_s)$. In ongoing work with O. Pikhurko, we are working towards pushing the lower bounds toward the upper bounds in the bipartite case. We have been successful in a few small cases thus far, see [25].

In joint work with R. Gould and T. Luczak ([16]), we have improved the upper bound previously determined in [1] for $sat(n, C_l)$, where $C_l$ denotes a cycle of length $l$. The upper bound follows via a construction, and we conjecture that our construction is asymptotically optimal for sufficiently large $n$ and $l$.

3 Future Work

In addition, the following problems will be part of my future research.

3.1 “Potential” Problems

Determining the value of $\sigma(F, n)$ when $F$ is an $r$-uniform hypergraph is also interesting, though may prove to be very difficult since there is no known Erdős-Gallai type criteria [8] for hypergraphs. We have again been able to obtain a lower bound, and our edge-swapping technique seems plausible in the $r = 3$ case as indicated by recent results of Kocay and Li.
3.2 Saturated Problems

Determine \( sat(n, K^s_t) \) for \( t, s \geq 2 \). While we have made significant progress towards this goal much remains to be done. Proving that \( sat(n, F, \delta) \) is a monotonically increasing function in \( \delta \) would be of much use. Duffus and Hanson [6] have made some progress in this regard. I am also interested in the hypercube analogue of this problem. That is, in an \( n \)-dimensional hypercube, \( Q_n \), determine the minimum number of edges a subgraph of \( Q_n \) must contain to be \( Q_m \)-saturated, where \( m < n \). With an undergraduate student, Anthony Santolupo, we have investigated this problem, yet most cases remain open.

3.3 The Unknown

In addition, I look forward to working with colleagues from my new setting and from around the world. I greatly enjoy working with other mathematicians, as is evidenced by my number of co-authors, and look forward to this continued opportunity. I also feel fortunate in having been able to attend many conferences (BCC, SIAM, S.E. Conf. on Graph Theory & Comp., and others), where I have been stimulated by mathematicians from around the world. I am eager to continue collaboration with friends I have made and students I supervise, and tackle problems I have encountered.

References


