Graphic Sequences with a Realization Containing a Generalized Friendship Graph *

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Abstract: Gould, Jacobson and Lehel (Combinatorics, Graph Theory and Algorithms, Vol.I (1999) 451–460) considered a variation of the classical Turán-type extremal problems as follows: for any simple graph H, determine the smallest even integer $\sigma(H,n)$ such that every n-term graphic sequence $\pi=(d_1,d_2,\ldots,d_n)$ with term sum $\sigma(\pi)=d_1+d_2+\cdots+d_n\geq\sigma(H,n)$ has a realization G containing H as a subgraph. Let $F_{t,r,k}$ denote the generalized friendship graph on $kt-kr+r$ vertices, that is, the graph of k copies of K_t meeting in a common r set, where K_t is the complete graph on t vertices and $0\leq r\leq t$. In this paper, we determine $\sigma(F_{t,r,k},n)$ for $k\geq 2$, $t\geq 3$, $1\leq r\leq t-2$ and n sufficiently large.

Keywords: degree sequence, potentially $F_{t,r,k}$-graphic sequence, generalized friendship graph.

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1. Introduction

The set of all sequences $\pi=(d_1,d_2,\ldots,d_n)$ of non-negative, non-increasing integers with $d_1\leq n-1$ is denoted by NS_n. A sequence $\pi\in NS_n$ is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The set of

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all graphic sequences in NS_n is denoted by GS_n. For a sequence $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, denote $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$. For a given graph H, a graphic sequence π is said to be *potentially* (respectively, *forcibly*) H*-graphic* if there exists a realization of π containing H as a subgraph (respectively, each realization of π contains H as a subgraph). Given any two graphs G and H, $G \cup H$ is the disjoint union of G and H and $G + H$, their join, is the graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{uv|u \in V(G), v \in V(H)\}$.

The classical problem in extremal graph theory is as follows: given a subgraph H determine the smallest integer m such that every graph G on n vertices with edge number $e(G) \geq m$ contains H as a subgraph. This m is denoted by $ex(H,n)$, and is called the *Turán number* of H. In terms of graphic sequences, the number $2ex(H,n)$ is the smallest even integer such that each sequence $\pi \in GS_n$ with $\sigma(\pi) \geq 2ex(H,n)$ is forcibly H-graphic. Gould, Jacobson and Lehel [7] considered the following variation of the classical Turán number $ex(H,n)$: determine the smallest even integer $\sigma(H,n)$ such that each sequence $\pi \in GS_n$ with $\sigma(\pi) \geq \sigma(H,n)$ is potentially H-graphic. The instance of this problem when $H = K_r$, the complete graph on r vertices, was considered by Erdős, Jacobson and Lehel [3] where they showed that $\sigma(K_3,n) = 2n$ for $n \geq 6$ and conjectured that $\sigma(K_r,n) = (r-2)(2n-r+1)+2$ for n sufficiently large. Gould et al. [7] and Li and Song [11] independently proved it for $r = 4$. In [12,13], Li, Song and Luo showed that the conjecture holds for $r = 5$ and $n \geq 10$ and for $r \geq 6$ and $n \geq \binom{r-1}{2} + 3$. Recently, Li and Yin [14] further determined $\sigma(K_r,n)$ for $r \geq 7$ and $n \geq 2r+1$. The problem of determining $\sigma(K_r,n)$ is completely solved.

For $0 \leq r \leq t$, denote the generalized friendship graph on $kt-kr+r$ vertices by $F_{t,r,k}$, where $F_{t,r,k}$ is the graph of k copies of K_t meeting in a common r set. Clearly, $F_{t,r,k} = K_r + kK_{t-r}$, where kK_{t-r} is the disjoint union of k copies of K_{t-r}. Since $F_{t,r,1} = F_{t,t,k} = K_t$, we have that $\sigma(F_{t,r,1},n) = \sigma(F_{t,t,k},n) = \sigma(K_t,n)$. The graph $F_{2,0,k} = kK_2$ was considered by Gould et al. in [7], where they determined that $\sigma(F_{2,0,k},n) = (k-1)(2n-k)+2$. The graph $F_{3,1,k}$, the friendship graph, was considered by Ferrara, Gould and Schmitt in [5], where they determined that $\sigma(F_{3,1,k},n) = k(2n-k-1)+2$ for $n \geq \frac{9}{2}k^2 + \frac{7}{2}k - \frac{1}{2}$. Lai [10] determined $\sigma(F_{3,1,2},n)$. The graph $F_{t,t-1,k}$, the $r_1 \times \cdots \times r_t$ complete t-partite graph with $r_1 = \cdots = r_{t-1} = 1$ and $r_t = k$, was considered by Yin and Chen in [16], where they determined that

$$\sigma(F_{t,t-1,k},n)=\begin{cases}(k+2t-5)n-(t-2)(k+t-2)+2 & \text{if } k \text{ or } n-t+1 \text{ is odd,}\\ (k+2t-5)n-(t-2)(k+t-2)+1 & \text{if } k \text{ and } n-t+1 \text{ are even}\end{cases}$$

for $n \geq 3t+2k^2+3k-6$. In fact, [16] also contains a proof of the conjecture of Erdős et al. as a consequence of this main result (the case of $k=1$). The graph $F_{t,0,k}=kK_t$ and the graph $F_{t,t-2,k}$ were considered by Ferrara in [4], where he determined that $\sigma(F_{t,0,k},n)=(kt-k-1)(2n-kt+k)+2$ for a sufficiently large choice of n and $\sigma(F_{t,t-2,k},n)=(t+k-3)(2n-t-k+2)+2$ for a sufficiently large choice of n. The purpose of this paper is to determine $\sigma(F_{t,r,k},n)$ for $k \geq 2$, $t \geq 3$, $1 \leq r \leq t-2$ and n sufficiently large. That is, we establish all remaining cases. The following is our main result.

Theorem 1.1 *Let $k \geq 2$, $t \geq 3$ and $1 \leq r \leq t-2$. Then there exists a positive integer $g(t,r,k)$ such that for all $n \geq g(t,r,k)$,*

$$\sigma(F_{t,r,k},n)=(n(t,r,k)-k-1)(2n-n(t,r,k)+k)+2, \tag{1}$$

where $n(t,r,k)=kt-kr+r$ is the order of $F_{t,r,k}$.

One can see that $\sigma(F_{t,r,k},n) \geq (n(t,r,k)-k-1)(2n-n(t,r,k)+k)+2$ by considering the graphic sequence

$$\pi=((n-1)^{n(t,r,k)-k-1},(n(t,r,k)-k-1)^{n-n(t,r,k)+k+1}),$$

which has degree sum

$$\begin{aligned}\sigma(\pi) &= (n(t,r,k)-k-1)(n-1)+(n-n(t,r,k)+k+1)(n(t,r,k)-k-1)\\ &= (n(t,r,k)-k-1)(2n-n(t,r,k)+k),\end{aligned}$$

where the symbol x^y in a sequence stands for y consecutive terms, each equal to x. This sequence is uniquely realized by $K_{n(t,r,k)-k-1}+\overline{K}_{n-n(t,r,k)+k+1}$. To see that $K_{n(t,r,k)-k-1}+\overline{K}_{n-n(t,r,k)+k+1}$ contains no copy of $F_{t,r,k}$ first notice that any $k+1$ vertices of $F_{t,r,k}$ must contain at least one edge. Now if $K_{n(t,r,k)-k-1}+\overline{K}_{n-n(t,r,k)+k+1}$ were to contain a copy of $F_{t,r,k}$ it must contain at least $k+1$ of its vertices from the subgraph $\overline{K}_{n-n(t,r,k)+k+1}$ of $K_{n(t,r,k)-k-1}+\overline{K}_{n-n(t,r,k)+k+1}$, however this subgraph does not contain an edge. This lower bound first appeared in [15] and can also be generated using the techniques in [6]. For $r = 1$, Chen et al. [1] determined the Turán number of $F_{t,1,k}$ as follows:

$$ex(F_{t,1,k}, n) = ex(K_t, n) + \begin{cases} k^2 - k + 1 & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k + 1 & \text{if } k \text{ even.} \end{cases}$$

The Turán number of $F_{t,r,k}$ in the more general case is unknown.

2. Useful Known Results

For $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, let $d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1}$ be the rearrangement in non-increasing order of $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$. Then $\pi' = (d'_1, d'_2, \ldots, d'_{n-1})$ is called the *residual sequence* of π. For example, let $\pi = (4, 3, 3, 2, 2, 2)$. By deleting $d_1 = 4$, and then reducing the first 4 remaining terms of π by one, we get the integer sequence $2, 2, 1, 1, 2$. Reorder the integer sequence $2, 2, 1, 1, 2$ in non-increasing order, we get that the residual sequence of π is $\pi' = (2, 2, 2, 1, 1)$. It is easy to see that if π' is graphic then so is π, since a realization G of π can be obtained from a realization G' of π' by adding a new vertex of degree d_1 to G' and joining it to the vertices whose degrees are reduced by one in going from π to π'. Each of the following results will be useful as we proceed with the proof of Theorem 1.1.

Theorem 2.1 [17,18] *Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, $x = d_1$ and $\sigma(\pi)$ be even. If there exists an integer n_1, $n_1 \leq n$ such that $d_{n_1} \geq y \geq 1$ and $n_1 \geq \frac{1}{y}\left\lfloor \frac{(x+y+1)^2}{4} \right\rfloor$, then π is graphic.*

Theorem 2.2 [7] *If $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ has a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.*

Theorem 2.3 [19] *Let $n \geq 2m + 2$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $d_{m+1} \geq m$. If $d_{2m+2} \geq m - 1$, then π is potentially K_{m+1}-graphic.*

Theorem 2.4 [4] *Let $H = K_{m_1} \cup \cdots \cup K_{m_k}$, where each $m_i \geq 2$. Then for a sufficiently large choice of n,*

$$\sigma(H, n) = (m - k - 1)(2n - m + k) + 2,$$

where $m = \sum_{i=1}^{k} m_i$.

3. Proof of Main Result

From here forward, let $k \geq 2$, $t \geq 3$, $1 \leq r \leq t-2$ and n be a sufficiently large integer. We begin the proof of Theorem 1.1 by showing that any graphic degree sequence with sum at least that given in Equation (1) has certain properties. In each part of the following lemma the proof follows by a contradiction to the degree sum.

Lemma 3.1 *Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq (n(t,r,k)-k-1)(2n-n(t,r,k)+k)+2$. Then*

(1) $d_r \geq n(t,r,k)-1$,

(2) $d_{n(t,r,k)} \geq n(t,r,k)-k-1$,

(3) $d_{n(t,r,k)-k+1} \geq n(t,r,k)-k$,

(4) If there is some ℓ, $0 \leq \ell \leq kt-kr-k-2$ such that $d_{r+\ell} \geq n(t,r,k)-1$ and $d_{r+\ell+1} \leq n(t,r,k)-2$, then $d_{n(t,r,k)} \geq n(t,r,k)-k$,

(5) $p(\pi) \geq \sqrt{\sigma(\pi)}$, where $p(\pi) = max\{i|d_i \geq 1\}$.

Proof. (1) If $d_r \leq n(t,r,k)-2$, then for n sufficiently large,

$$\begin{aligned} \sigma(\pi) &\leq (r-1)(n-1)+(n-r+1)(n(t,r,k)-2) \\ &< (n(t,r,k)-k-1)(2n-n(t,r,k)+k)+2. \end{aligned}$$

(2) If $d_{n(t,r,k)} \leq n(t,r,k)-k-2$, then by applying the well-known Erdős-Gallai [2] characterization of degree sequences,

$$\begin{aligned} \sigma(\pi) &= \textstyle\sum_{i=1}^{n(t,r,k)-1} d_i + \sum_{i=n(t,r,k)}^{n} d_i \\ &\leq (n(t,r,k)-1)(n(t,r,k)-2) + \textstyle\sum_{i=n(t,r,k)}^{n} \min\{n(t,r,k)-1, d_i\}) \\ &\quad + \textstyle\sum_{i=n(t,r,k)}^{n} d_i \\ &= (n(t,r,k)-1)(n(t,r,k)-2) + 2\textstyle\sum_{i=n(t,r,k)}^{n} d_i \\ &\leq (n(t,r,k)-1)(n(t,r,k)-2) \\ &\quad +2(n-(n(t,r,k)-1))(n(t,r,k)-k-2) \\ &< (n(t,r,k)-k-1)(2n-n(t,r,k)+k)+2 \text{ for } n \text{ sufficiently large.} \end{aligned}$$

(3) If $d_{n(t,r,k)-k+1} \leq n(t,r,k)-k-1$, then as in the proof of part (2) we apply the result of Erdős-Gallai to reach a contradiction.

(4) If $d_{n(t,r,k)} \leq n(t,r,k)-k-1$, then

$$\begin{aligned}\sigma(\pi) &\leq (n-1)(r+\ell)+(n(t,r,k)-2)(kt-kr-\ell-1)\\ &\quad +(n(t,r,k)-k-1)(n-(n(t,r,k)-1))\\ &\leq (2n(t,r,k)-2k-3)n+(n(t,r,k)-2)(kt-kr-1)\\ &\quad -(n(t,r,k)-k-1)(n(t,r,k)-1)\\ &< (n(t,r,k)-k-1)(2n-n(t,r,k)+k)+2 \text{ for } n \text{ sufficiently large.}\end{aligned}$$

(5) Since $(p(\pi))^2 \geq p(\pi)(p(\pi)-1) \geq p(\pi)d_1 \geq \sum_{i=1}^{n} d_i = \sigma(\pi)$, we have $p(\pi) \geq \sqrt{\sigma(\pi)}$. □

Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with

$$n-2 \geq d_1 \geq \cdots \geq d_{n(t,r,k)} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n$$

and

$$\sigma(\pi) \geq (n(t,r,k)-k-1)(2n-n(t,r,k)+k)+2.$$

By Lemma 3.1, $d_r \geq n(t,r,k)-1$ and $d_{n(t,r,k)} \geq n(t,r,k)-k-1$. We construct the sequence

$$\pi_1 = (d_2^{(1)}, \ldots, d_{n(t,r,k)}^{(1)}, d_{n(t,r,k)+1}^{(1)}, \ldots, d_n^{(1)})$$

from π by deleting d_1, reducing the first d_1 remaining terms of π by one, and then reordering the last $n-n(t,r,k)$ terms to be non-increasing. For $2 \leq i \leq r$, we construct

$$\pi_i = (d_{i+1}^{(i)}, \ldots, d_{n(t,r,k)}^{(i)}, d_{n(t,r,k)+1}^{(i)}, \ldots, d_n^{(i)})$$

from

$$\pi_{i-1} = (d_i^{(i-1)}, \ldots, d_{n(t,r,k)}^{(i-1)}, d_{n(t,r,k)+1}^{(i-1)}, \ldots, d_n^{(i-1)})$$

by deleting $d_i^{(i-1)}$, reducing the first $d_i^{(i-1)}$ nonzero remaining terms of π_{i-1} by one, and then reordering the last $n-n(t,r,k)$ terms to be non-increasing. The manner in which we construct $\pi_i,\ r+1 \leq i \leq n(t,r,k)$ depends on two cases.

Case 1. $d_{n(t,r,k)-k-1} \geq n(t,r,k)-1$.

In this case, we proceed as above and construct $\pi_i,\ r+1 \leq i \leq n(t,r,k)$ from π_{i-1} by removing $d_i^{(i-1)}$, reducing the first $d_i^{(i-1)}$ nonzero remaining terms of π_{i-1} by one, and then reordering the last $n-n(t,r,k)$ terms to be non-increasing.

Case 2. There is some ℓ, $0 \le \ell \le kt-kr-k-2$ such that $d_{r+\ell} \ge n(t,r,k)-1$ and $d_{r+\ell+1} \le n(t,r,k)-2$.

By Lemma 3.1, $d_{n(t,r,k)} \ge n(t,r,k)-k$. In this case, we first construct π_i, $r+1 \le i \le r+\ell$ as above, by removing $d_i^{(i-1)}$ from π_{i-1}, reducing the first $d_i^{(i-1)}$ nonzero remaining terms of π_{i-1} by one, and then reordering the last $n-n(t,r,k)$ terms to be non-increasing. From the definition of π_i for $1 \le i \le r+\ell$, it is easy to see that

$$\begin{aligned}\pi_{r+\ell} &= (d_{r+\ell+1}^{(r+\ell)}, \ldots, d_{n(t,r,k)}^{(r+\ell)}, d_{n(t,r,k)+1}^{(r+\ell)}, \ldots, d_n^{(r+\ell)}) \\ &= (d_{r+\ell+1}-(r+\ell), \ldots, d_{n(t,r,k)}-(r+\ell), d_{n(t,r,k)+1}^{(r+\ell)}, \ldots, d_n^{(r+\ell)}),\end{aligned}$$

and hence

$$\begin{aligned}kt-kr-\ell-2 \ge d_{r+\ell+1}^{(r+\ell)} \ge \cdots \ge d_{n(t,r,k)}^{(r+\ell)} &\ge kt-kr-\ell-k \\ &= k(t-r-\tfrac{\ell}{k}-1) \\ &= (t-r-\tfrac{\ell}{k}-1)+(k-1)(t-r-\tfrac{\ell}{k}-1) \\ &\ge (t-r-\tfrac{\ell}{k}-1)+(k-1)(t-r-\tfrac{kt-kr-k-2}{k}-1) \\ &\ge t-r-\tfrac{\ell}{k}.\end{aligned}$$

Moreover,

$$d_{n(t,r,k)+1}^{(r+\ell)} \ge d_{n(t,r,k)+1}-(r+\ell) \ge kt-kr-\ell-k \ge 2$$

and the terms $d_{n(t,r,k)+1}^{(r+\ell)}, \ldots, d_{d_1+2}^{(r+\ell)}$ differ by at most one. This implies that $p(\pi_{r+\ell}) = p(\pi)$. Let $\ell = \ell_1+\ell_2+\cdots+\ell_k$, where $\ell_i = \lfloor \frac{\ell}{k} \rfloor$ or $\lfloor \frac{\ell}{k} \rfloor + 1$ for each $i = 1, \ldots, k$. In other words, ℓ is partitioned into k parts of sizes $\ell_1, \ldots, \ell_k$ as evenly as possible. Denote $x_i = t-r-\ell_i$ for each $i = 1, \ldots, k$. Then for each $i = 1, \ldots, k$, we have

$$kt-kr-\ell-2 \ge d_{r+\ell+1}^{(r+\ell)} \ge \cdots \ge d_{n(t,r,k)}^{(r+\ell)} \ge x_i \ge 1.$$

Let

$$\begin{array}{l} d_{r+\ell+j}^{(r+\ell)} = f_{r+\ell+j} + (x_1-1) \text{ for } 1 \le j \le x_1, \\ d_{r+\ell+x_1+j}^{(r+\ell)} = f_{r+\ell+x_1+j} + (x_2-1) \text{ for } 1 \le j \le x_2, \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ d_{r+\ell+x_1+\cdots+x_{k-1}+j}^{(r+\ell)} = f_{r+\ell+x_1+\cdots+x_{k-1}+j} + (x_k-1) \text{ for } 1 \le j \le x_k. \end{array}$$

Clearly, $1 \le f_{r+\ell+m} \le (kt-kr-\ell-2)-(t-r-\lfloor \frac{\ell}{k} \rfloor)+2$ for each $m = 1, \ldots, kt-kr-\ell$. We now construct π_i, $r+\ell+1 \le i \le n(t,r,k)$ from π_{i-1} by removing $d_i^{(i-1)}$, reducing the first f_i

nonzero terms, *starting with* $d^{(i-1)}_{n(t,r,k)+1}$ by one, and then ordering the last $n-n(t,r,k)$ terms to be non-increasing. Note that if n is sufficiently large, then $p(\pi_{r+\ell}) = p(\pi) \geq \sqrt{\sigma(\pi)}$ (by Lemma 3.1) is also sufficiently large. Moreover, $f_{r+\ell+m} \leq kt-kr$ for each $m = 1,\ldots,kt-kr-\ell$. Thus, we can be assured that for n large enough, there is a sufficient number of positive terms in each π_{i-1} $(r+\ell+1 \leq i \leq n(t,r,k))$ to construct π_i without forcing any terms in π_i to be negative. We now present the following crucial lemma.

Lemma 3.2 *If $\pi_{n(t,r,k)}$ is graphic, then π is potentially $F_{t,r,k}$-graphic (i.e. potentially $K_r + kK_{t-r}$)-graphic.*

Proof. Let $G_{n(t,r,k)}$ be a realization of $\pi_{n(t,r,k)}$ with $V(G_{n(t,r,k)}) = \{v_{n(t,r,k)+1},\ldots,v_n\}$ and $d(v_{n(t,r,k)+j}) = d^{(n(t,r,k))}_{n(t,r,k)+j}$ for $1 \leq j \leq n-n(t,r,k)$, where $d(v_{n(t,r,k)+j})$ is the degree of $v_{n(t,r,k)+j}$ in $G_{n(t,r,k)}$. Denote $\pi_0 = \pi$. The proof of Lemma 3.2 now breaks into the following two cases.

Case 1. $d_{n(t,r,k)-k-1} \geq n(t,r,k)-1$.

For $i = n(t,r,k)-1,\ldots,1,0$ in turn, we can construct a realization G_i of π_i from the realization G_{i+1} of π_{i+1} by adding a new vertex v_{i+1} to G_{i+1} and joining it to the vertices whose degrees were reduced by one in going from π_i to π_{i+1}. Since $d_{n(t,r,k)-k+1} \geq n(t,r,k)-k$ (by Lemma 3.1), we have $d^{(n(t,r,k)-k-1)}_{n(t,r,k)-k} \geq d^{(n(t,r,k)-k-1)}_{n(t,r,k)-k+1} \geq 1$, and hence $v_{n(t,r,k)-k}v_{n(t,r,k)-k+1} \in E(G_{n(t,r,k)-k-1})$. In creating $\pi_1,\ldots,\pi_{n(t,r,k)-k-1}$, the fact that $d_{n(t,r,k)-k-1} \geq n(t,r,k)-1$ implies that the realization G_0 of π created in this manner will contain a copy of $K_r + (K_{kt-kr-k-1} + (K_2 \cup (k-1)K_1))$ such that $V(K_r) = \{v_1,\ldots,v_r\}$, $V(K_{kt-kr-k-1}) = \{v_{r+1},\ldots,v_{n(t,r,k)-k-1}\}$, $V(K_2) = \{v_{n(t,r,k)-k}, v_{n(t,r,k)-k+1}\}$ and $V((k-1)K_1) = \{v_{n(t,r,k)-k+2},\ldots,v_{n(t,r,k)}\}$. It is easy to see that $K_{kt-kr-k-1} + (K_2 \cup (k-1)K_1)$ contains kK_{t-r} as a subgraph. Thus, G_0 contains $K_r + kK_{t-r}$ as a subgraph.

Case 2. There is some ℓ, $0 \leq \ell \leq kt-kr-k-2$ such that $d_{r+\ell} \geq n(t,r,k)-1$ and $d_{r+\ell+1} \leq n(t,r,k)-2$.

For $i = n(t,r,k)-1,\ldots,r+\ell+1,r+\ell$ in turn, we can construct G_i from G_{i+1} by adding a new vertex v_{i+1} to G_{i+1} and joining it to vertices of those degrees that were reduced by one in the formation of π_{i+1}. It is easy to see that $G_{r+\ell}$ is a realization of

$$(f_{r+\ell+1},\ldots,f_{n(t,r,k)}, d^{(r+\ell)}_{n(t,r,k)+1},\ldots,d^{(r+\ell)}_n)$$

such that $d(v_{r+\ell+j}) = f_{r+\ell+j}$ for $1 \leq j \leq kt - kr - \ell$ and $\{v_{r+\ell+1}, \ldots, v_{n(t,r,k)}\}$ forms an independent set in $G_{r+\ell}$. We now construct a realization $G'_{r+\ell}$ of $\pi_{r+\ell}$ from $G_{r+\ell}$ by adding those edges such that $\{v_{r+\ell+x_0+\cdots+x_{j-1}+1}, \ldots, v_{r+\ell+x_0+\cdots+x_{j-1}+x_j}\}$ forms a clique for each $j = 1, \ldots, k$, where $x_0 = 0$. For convenience, the graph $G'_{r+\ell}$ is still denoted by $G_{r+\ell}$. For $i = r+\ell-1, \ldots, 1, 0$ in turn, we then can construct a realization G_i of π_i from the realization G_{i+1} of π_{i+1} by adding a new vertex v_{i+1} to G_{i+1} and joining it to the vertices whose degrees were reduced by one in going from π_i to π_{i+1}. The fact that $d_{r+\ell} \geq n(t, r, k) - 1$ implies that G_0 contains $K_r + (K_\ell + (K_{x_1} \cup \cdots \cup K_{x_k}))$ as a subgraph. It is easy to see that $K_\ell + (K_{x_1} \cup \cdots \cup K_{x_k})$ contains kK_{t-r} as a subgraph. Therefore, G_0 contains $K_r + kK_{t-r}$ as a subgraph. $\Box$

Proof of Theorem 1.1. In order to prove

$$\sigma(F_{t,r,k}, n) \leq (n(t, r, k) - k - 1)(2n - n(t, r, k) + k) + 2,$$

it is enough to prove that if $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with

$$\sigma(\pi) \geq (n(t, r, k) - k - 1)(2n - n(t, r, k) + k) + 2,$$

then π is potentially $F_{t,r,k}$-graphic (i.e. potentially $K_r + kK_{t-r}$-graphic). The proof follows by induction on r (and any $t \geq r+2$). If $r = 0$, then $\sigma(\pi) \geq (kt-k-1)(2n-kt+k)+2$. By Theorem 2.4 (the case of $m_1 = \cdots = m_k = t$), π is potentially $F_{t,0,k}$ -graphic (i.e. potentially kK_t-graphic) for any $t \geq 2$. Now we assume that the result holds for $r - 1$ (and any $t \geq (r - 1) + 2$), where $r \geq 1$. We will prove that the result holds for r (and any $t \geq r + 2$). Let $\pi' = (d'_1, d'_2, \ldots, d'_{n-1})$ be the residual sequence of π. By the well-known result of Havel [9] and Hakimi [8], π' is graphic and

$$\begin{aligned}
\sigma(\pi') &= \sigma(\pi) - 2d_1 \\
&\geq (n(t, r, k) - k - 1)(2n - n(t, r, k) + k) + 2 - 2(n - 1) \\
&= (n(t - 1, r - 1, k) - k - 1)(2(n - 1) - n(t - 1, r - 1, k) + k) + 2.
\end{aligned}$$

By $t - 1 \geq (r - 1) + 2$ and the induction hypothesis, π' is potentially $F_{t-1,r-1,k}$ -graphic (i.e. potentially $K_{r-1} + kK_{(t-1)-(r-1)}$-graphic). In other words, there is some realization of π' that contains a copy of $K_{r-1} + kK_{(t-1)-(r-1)}$. Furthermore, by Theorem 2.2, this implies that there exists a realization of π' with $K_{r-1} + kK_{(t-1)-(r-1)} = K_{r-1} + kK_{t-r}$ on those vertices having degree $d'_1, d'_2, \ldots, d'_{n(t,r,k)-1}$. Now suppose that either $d_1 = n - 1$ or there exists an integer h,

$n(t,r,k) \leq h \leq d_1+1$ such that $d_h > d_{h+1}$. Then $d'_i = d_{i+1}-1$ for each $i = 1,\ldots,n(t,r,k)-1$. This implies that π would be potentially $K_r + kK_{t-r}$-graphic. Thus, we may assume that no such h exists and hence that

$$n-2 \geq d_1 \geq \cdots \geq d_{n(t,r,k)} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n.$$

If $d_{2n(t,r,k)} \geq n(t,r,k)-1$, then π is potentially $K_{n(t,r,k)}$-graphic by Theorem 2.3, which is sufficient to show that π is potentially $K_r + kK_{t-r}$-graphic. We now may further assume that $d_{2n(t,r,k)} \leq n(t,r,k)-2$. If $d_1 \leq 2n(t,r,k)-3$, then

$$\sigma(\pi) \leq (2n(t,r,k)-3)(2n(t,r,k)-1) + (n(t,r,k)-2)(n-(2n(t,r,k)-1)).$$

This is less than $(n(t,r,k)-k-1)(2n-n(t,r,k)+k)+2$ for n sufficiently large. Hence $d_1 \geq 2n(t,r,k)-2$, i.e., $d_1+2 \geq n(t,r,k)$. This implies that

$$n(t,r,k)-2 \geq d_{n(t,r,k)} = d_{n(t,r,k)+1} = \cdots = d_{2n(t,r,k)} = \cdots = d_{d_1+2}.$$

For each $j = 1,\ldots,n(t,r,k)$, the terms $d^{(j)}_{n(t,r,k)+1},\ldots,d^{(j)}_{2n(t,r,k)},\ldots,d^{(j)}_{d_1+2}$ differ by at most one. Hence $\pi_{n(t,r,k)}$ satisfies, for some $x \geq 1$,

$$n(t,r,k)-2 \geq x = d^{(n(t,r,k))}_{n(t,r,k)+1} \geq \cdots \geq d^{(n(t,r,k))}_{2n(t,r,k)} \geq \cdots \geq d^{(n(t,r,k))}_{d_1+2} \geq x-1.$$

If $x = 1$, $\pi_{n(t,r,k)}$ must be graphic as $\sigma(\pi_{n(t,r,k)})$ is even. If $x \geq 2$, then

$$\frac{1}{x-1}\left\lfloor \frac{(x+(x-1)+1)^2}{4} \right\rfloor \leq x+2 \leq n(t,r,k).$$

By Theorem 2.1, $\pi_{n(t,r,k)}$ is also graphic. Thus, π is potentially $K_r + kK_{t-r}$-graphic by Lemma 3.2. $\square$

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