Graphic Sequences with a Realization Containing a
Generalized Friendship Graph

Jian-Hua Yin\textsuperscript{a}\dagger, Gang Chen\textsuperscript{b}, John R. Schmitt\textsuperscript{c}

\textsuperscript{a}Department of Applied Mathematics, College of Information Science and Technology,
Hainan University, Haikou 570228, P.R. China

\textsuperscript{b}Department of Mathematics, Ningxia University, Yinchuan 750021, P.R. China

\textsuperscript{c}Department of Mathematics, Middlebury College, Middlebury, VT, USA

Abstract: Gould, Jacobson and Lehel (Combinatorics, Graph Theory and Algorithms, Vol.I (1999) 451–460) considered a variation of the classical Turán-type extremal problems as follows: for any simple graph $H$, determine the smallest even integer $\sigma(H, n)$ such that every $n$-term graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ with term sum $\sigma(\pi) = d_1 + d_2 + \cdots + d_n \geq \sigma(H, n)$ has a realization $G$ containing $H$ as a subgraph. Let $F_{t, r, k}$ denote the generalized friendship graph on $kt - kr + r$ vertices, that is, the graph of $k$ copies of $K_t$ meeting in a common $r$ set, where $K_t$ is the complete graph on $t$ vertices and $0 \leq r \leq t$. In this paper, we determine $\sigma(F_{t, r, k}, n)$ for $k \geq 2$, $t \geq 3$, $1 \leq r \leq t - 2$ and $n$ sufficiently large.

Keywords: degree sequence, potentially $F_{t, r, k}$-graphic sequence, generalized friendship graph.

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1. Introduction

The set of all sequences $\pi = (d_1, d_2, \ldots, d_n)$ of non-negative, non-increasing integers with $d_1 \leq n - 1$ is denoted by $NS_n$. A sequence $\pi \in NS_n$ is said to be graphic if it is the degree sequence of a simple graph $G$ on $n$ vertices, and such a graph $G$ is called a realization of $\pi$. The set of

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\textsuperscript{†}E-mail: yinjh@ustc.edu
all graphic sequences in $NS_n$ is denoted by $GS_n$. For a sequence $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, denote $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$. For a given graph $H$, a sequence $\pi \in GS_n$ is said to be potentially $H$-graphic if there exists a realization of $\pi$ containing $H$ as a subgraph. Given any two graphs $G$ and $H$, $G \cup H$ is the disjoint union of $G$ and $H$ and $G + H$, their join, is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv|u \in V(G), v \in V(H)\}$.

Gould, Jacobson and Lehel [7] considered the following variation of the classical Turán-type extremal problems: determine the smallest even integer $\sigma(H, n)$ such that each sequence $\pi \in GS_n$ with $\sigma(\pi) \geq \sigma(H, n)$ is potentially $H$-graphic. If $H = K_r$, the complete graph on $r$ vertices, this problem was considered by Erdős, Jacobson and Lehel [3] where they showed that $\sigma(K_3, n) = 2n$ for $n \geq 6$ and conjectured that $\sigma(K_r, n) = (r-2)(2n-r+1)+2$ for $n$ sufficiently large. Gould et al. [7] and Li and Song [10] independently proved it for $r = 4$. In [11,12], Li, Song and Luo showed that the conjecture holds for $r = 5$ and $n \geq 10$ and for $r \geq 6$ and $n \geq \left(\binom{r-1}{2}\right)+3$. Recently, Li and Yin [13] further determined $\sigma(K_r, n)$ for $r \geq 7$ and $n \geq 2r+1$. The purpose of this paper is to determine $\sigma(K_r, n)$ was completely solved.

For $0 \leq r \leq t$, denote the generalized friendship graph on $kt - kr + r$ vertices by $F_{t,r,k}$, where $F_{t,r,k}$ is the graph of $k$ copies of $K_t$ meeting in a common $r$ set. Clearly, $F_{t,r,k} = K_r + kK_{t-r}$, where $kK_{t-r}$ is the disjoint union of $k$ copies of $K_{t-r}$. Since $F_{t,1,1} = F_{t,1,k} = K_t$, we have that $\sigma(F_{t,1,1}, n) = \sigma(F_{t,1,k}, n) = \sigma(K_t, n)$. The graph $F_{2,0,k} = kK_2$ was considered by Gould et al. in [7], where they determined that $\sigma(F_{2,0,k}, n) = (k-1)(2n-k) + 2$. The graph $F_{3,1,k}$, the friendship graph, was considered by Ferrara, Gould and Schmitt in [5], where they determined that $\sigma(F_{3,1,k}, n) = k(2n-k-1) + 2$ for $n \geq \frac{9}{2}k^2 + \frac{7}{2}k - \frac{1}{2}$. The graph $F_{t,t-1,k}$, the $r_1 \times \cdots \times r_t$ complete $t$-partite graph with $r_1 = \cdots = r_{t-1} = 1$ and $r_t = k$, was considered by Yin and Chen in [15], where they determined that

$$\sigma(F_{t,t-1,k}, n) = \begin{cases} (k+2t-5)n-(t-2)(k+t-2) + 2 & \text{if } k \text{ or } n-t+1 \text{ is odd}, \\ (k+2t-5)n-(t-2)(k+t-2) + 1 & \text{if } k \text{ and } n-t+1 \text{ are even} \end{cases}$$

for $n \geq 3t+2k^2+3k-6$. The graph $F_{t,0,k} = kK_t$ and the graph $F_{t,t-2,k}$ were considered by Ferrara in [4], where they determined that $\sigma(F_{t,0,k}, n) = (kt-k-1)(2n-kt+k) + 2$ for a sufficiently large choice of $n$ and $\sigma(F_{t,t-2,k}, n) = (t+k-3)(2n-t-k+2) + 2$ for a sufficiently large choice of $n$. The purpose of this paper is to determine $\sigma(F_{t,r,k}, n)$ for $k \geq 2$, $t \geq 3$,
$1 \leq r \leq t - 2$ and $n$ sufficiently large. That is, we establish all remaining cases. The following is our main result.

**Theorem 1.1** Let $k \geq 2$, $t \geq 3$ and $1 \leq r \leq t - 2$. Then there exists a positive integer $n(t, r, k)$ such that for all $n \geq n(t, r, k)$,$$
(kt - kr - k + r - 1)(2n - kt + kr + k - r) + 2.
$$

One can see that $\sigma(F_{t,r,k}, n) \geq (kt - kr - k + r - 1)(2n - kt + kr + k - r) + 2$ by considering the graphic sequence

$$
\pi = \left( (n - 1)^{kt - kr - k + r - 1}, (kt - kr - k + r - 1)^{n - kt + kr + k - r + 1} \right),
$$

which has degree sum

$$
\sigma(\pi) = (kt - kr - k + r - 1)(n - 1) + (n - kt + kr + k - r + 1)(kt - kr - k + r - 1)
= (kt - kr - k + r - 1)(2n - kt + kr + k - r).
$$

This sequence is uniquely realized by $K_{kt - kr - k + r - 1} + K_{n - kt + kr + k - r + 1}$, where the symbol $x^y$ in a sequence stands for $y$ consecutive terms, each equal to $x$. To see that $K_{kt - kr - k + r - 1} + K_{n - kt + kr + k - r + 1}$ contains no copy of $F_{t,r,k}$ first notice that any $k + 1$ vertices of $F_{t,r,k}$ must contain at least one edge. Now if $K_{kt - kr - k + r - 1} + K_{n - kt + kr + k - r + 1}$ were to contain a copy of $F_{t,r,k}$ it must contain at least $k + 1$ of its vertices from the subgraph $K_{n - kt + kr + k - r + 1}$ of $K_{kt - kr - k + r - 1} + K_{n - kt + kr + k - r + 1}$, however this subgraph does not contain an edge. This lower bound first appeared in [14] and can also be generated using the techniques in [6]. The instance of our result with $r = 1$ is an analogue to an extremal result of Guantao Chen et al. [1] which states that any graph on $n$ vertices having at least

$$
ex(n, K_t) + \begin{cases} 
  \begin{align*}
  k^2 - k + 1 & \text{ if } k \text{ is odd,} \\
  \frac{3}{2}k^2 + 1 & \text{ if } k \text{ even}
  \end{align*}
\end{cases}
$$

edges contains a copy of $F_{t,1,k}$. The number of edges necessary to guarantee a copy of $F_{t,r,k}$ in the more general case is unknown.
2. Useful Known Results

For $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, let $d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1}$ be the rearrangement in non-increasing order of $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$. Then $\pi' = (d'_1, d'_2, \ldots, d'_{n-1})$ is called the residual sequence of $\pi$. It is easy to see that if $\pi'$ is graphic then so is $\pi$, since a realization $G$ of $\pi$ can be obtained from a realization $G'$ of $\pi'$ by adding a new vertex of degree $d_1$ to $G'$ and joining it to the vertices whose degrees are reduced by one in going from $\pi$ to $\pi'$.

**Theorem 2.1** [8,9] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$. Then $\pi$ is graphic if and only if $\pi'$ is graphic.

**Theorem 2.2** [2] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ with even $\sigma(\pi)$. Then $\pi$ is graphic if and only if for any $h$, $1 \leq h \leq n - 1$,
\[
\sum_{i=1}^{h} d_i \leq h(h-1) + \sum_{j=h+1}^{n} \min\{h, d_j\}.
\]

**Theorem 2.3** [16,17] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, $x = d_1$ and $\sigma(\pi)$ be even. If there exists an integer $n_1$, $n_1 \leq n$ such that $d_{n_1} \geq y \geq 1$ and $n_1 \geq \left\lfloor \frac{(x+y+1)^2}{4} \right\rfloor$, then $\pi$ is graphic.

**Theorem 2.4** [7] If $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ has a realization $G$ containing $H$ as a subgraph, then there exists a realization $G'$ of $\pi$ containing $H$ as a subgraph so that the vertices of $H$ have the largest degrees of $\pi$.

**Theorem 2.5** [18] Let $n \geq 2m + 2$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $d_{m+1} \geq m$. If $d_{2m+2} \geq m - 1$, then $\pi$ is potentially $K_{m+1}$-graphic.

**Theorem 2.6** [4] Let $H = K_{m_1} \cup \cdots \cup K_{m_k}$, where each $m_i \geq 2$. Then for a sufficiently large choice of $n$,
\[
\sigma(H, n) = (m - k - 1)(2n - m + k) + 2,
\]
where $m = \sum_{i=1}^{k} m_i$.

3. Proof of Main Result

From here forward, let $k \geq 2$, $t \geq 3$, $1 \leq r \leq t - 2$ and $n$ be a sufficiently large integer. We begin the proof of Theorem 1.1 by showing that any graphic degree sequence with sum at least
that given in Equation (1) has certain properties. In each part of the following lemma the proof follows by a contradiction to the degree sum.

Lemma 3.1 Let \( \pi = (d_1, d_2, \ldots, d_n) \in GS_n \) with \( \sigma(\pi) \geq (kt - kr - k + r - 1)(2n - kt + kr + k - r) + 2 \). Then

(1) \( d_r \geq kt - kr + r - 1 \),
(2) \( d_{kt - kr + r} \geq kt - kr + r - k - 1 \),
(3) \( d_{kt - kr + r - k + 1} \geq kt - kr + r - k \),
(4) If there is some \( \ell, 0 \leq \ell \leq kt - kr - k - 2 \) such that \( d_{r+\ell} \geq kt - kr + r - 1 \) and \( d_{r+\ell+1} \leq kt - kr + r - 2 \), then \( d_{kt - kr + r} \geq kt - kr + r - k \),
(5) \( p(\pi) \geq \sqrt{\sigma(\pi)} \), where \( p(\pi) = \max\{i|d_i \geq 1\} \).

Proof. (1) If \( d_r \leq kt - kr + r - 2 \), then for \( n \) sufficiently large,

\[
\sigma(\pi) \leq (r - 1)(n - 1) + (n - r + 1)(kt - kr + r - 2) < (kt - kr - k + r - 1)(2n - kt + kr + k - r) + 2.
\]

(2) If \( d_{kt - kr + r} \leq kt - kr + r - k - 2 \), then by Theorem 2.2,

\[
\sigma(\pi) = \sum_{i=1}^{kt - kr + r - 1} d_i + \sum_{i=kt - kr + r}^{n} d_i \\
\leq ((kt - kr + r - 1)(kt - kr + r - 2) + \sum_{i=kt - kr + r}^{n} \min\{kt - kr + r - 1, d_i\}) \\
+ \sum_{i=kt - kr + r}^{n} d_i \\
= (kt - kr + r - 1)(kt - kr + r - 2) + 2 \sum_{i=kt - kr + r}^{n} d_i \\
\leq (kt - kr + r - 1)(kt - kr + r - 2) \\
+ 2(n - (kt - kr + r - 1))(kt - kr + r - k - 2) \\
< (kt - kr - k + r - 1)(2n - kt + kr + k - r) + 2 \text{ for } n \text{ sufficiently large.}
\]

(3) If \( d_{kt - kr + r - k + 1} \leq kt - kr + r - k - 1 \), then as in the proof of part (2) we apply Theorem 2.2 to reach a contradiction.
(4) If \( d_{kt-kr+r} \leq kt - kr + r - k - 1 \), then

\[
\sigma(\pi) \leq (n - 1)(r + \ell) + (kt - kr + r - 2)(kt - kr - \ell - 1) + (kt - kr + r - k - 1)(n - (kt - kr + r - 1)) \\
\leq (2kt - 2kr - 2k + 2r - 3)n + (kt - kr + r - 2)(kt - kr - 1) + (kt - kr + r - k - 1)(kt - kr + r - 1) \\
< (kt - kr - k + r - 1)(2n - kt + kr + k - r) + 2 \text{ for } n \text{ sufficiently large.}
\]

(5) Since \( (p(\pi))^2 \geq p(\pi)(p(\pi) - 1) \geq p(\pi)d_1 \geq \sum_{i=1}^{n} d_i = \sigma(\pi) \), we have \( p(\pi) \geq \sqrt{\sigma(\pi)} \). □

Let \( \pi = (d_1, d_2, \ldots, d_n) \in GS_n \) with

\[
n - 2 \geq d_1 \geq \cdots \geq d_{kt-kr+r} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n
\]

and

\[
\sigma(\pi) \geq (kt - kr - k + r - 1)(2n - kt + kr + k - r) + 2.
\]

By Lemma 3.1, \( d_r \geq kt - kr + r - 1 \) and \( d_{kt-kr+r} \geq kt - kr + r - k - 1 \). We construct the sequence

\[
\pi_1 = (d_2^{(1)}, \ldots, d_{kt-kr+r}^{(1)}, d_{kt-kr+r+1}^{(1)}, \ldots, d_n^{(1)})
\]

from \( \pi \) by deleting \( d_1 \), reducing the first \( d_1 \) remaining terms of \( \pi \) by one, and then reordering the last \( n - (kt - kr + r) \) terms to be non-increasing. For \( 2 \leq i \leq r \), we construct

\[
\pi_i = (d_{i+1}^{(i)}, \ldots, d_{kt-kr+r}^{(i)}, d_{kt-kr+r+1}^{(i)}, \ldots, d_n^{(i)})
\]

from

\[
\pi_{i-1} = (d_i^{(i-1)}, \ldots, d_{kt-kr+r}^{(i-1)}, d_{kt-kr+r+1}^{(i-1)}, \ldots, d_n^{(i-1)})
\]

by deleting \( d_i^{(i-1)} \), reducing the first \( d_i^{(i-1)} \) nonzero remaining terms of \( \pi_{i-1} \) by one, and then reordering the last \( n - (kt - kr + r) \) terms to be non-increasing. The manner in which we construct \( \pi_i \), \( r + 1 \leq i \leq kt - kr + r \) depends on two cases.

**Case 1.** \( d_{kt-kr+r-k-1} \geq kt - kr + r - 1 \).

In this case, we proceed as above and construct \( \pi_i \), \( r + 1 \leq i \leq kt - kr + r \) from \( \pi_{i-1} \) by removing \( d_i^{(i-1)} \), reducing the first \( d_i^{(i-1)} \) nonzero remaining terms of \( \pi_{i-1} \) by one, and then
reordering the last \(n - (kt - kr + r)\) terms to be non-increasing.

**Case 2.** There is some \(\ell, \ 0 \leq \ell \leq kt - kr - k - 2\) such that \(d_{r+\ell} \geq kt - kr + r - 1\) and \(d_{r+\ell+1} \leq kt - kr + r - 2\). By Lemma 3.1, \(d_{kt - kr + r} \geq kt - kr + r - k\).

In this case, we first construct \(\pi_i, \ r + 1 \leq i \leq r + \ell\) as above, by removing \(d^{(i-1)}_i\) from \(\pi_{i-1}\), reducing the first \(d^{(i-1)}_i\) nonzero remaining terms of \(\pi_{i-1}\) by one, and then reordering the last \(n - (kt - kr + r)\) terms to be non-increasing. From the definition of \(\pi_i\) for \(1 \leq i \leq r + \ell\), it is easy to see that

\[
\pi_{r+\ell} = (d^{(r+\ell)}_{r+\ell+1}, \ldots, d^{(r+\ell)}_{kt - kr + r}, d^{(r+\ell)}_{kt - kr + r+1}, \ldots, d^{(r+\ell)}_n)
\]

and hence

\[
kt - kr - \ell - 2 \geq d^{(r+\ell)}_{r+\ell+1} \geq \cdots \geq d^{(r+\ell)}_{kt - kr + r} \geq \begin{cases} kt - kr - \ell - k & \text{if } \ell = \ell_1 + \ell_2 + \cdots + \ell_k \text{ with } \ell_i = \lfloor \frac{\ell}{k} \rfloor + 1 \text{ for each } i = 1, \ldots, k, \\ t - r - \ell & \text{otherwise} \end{cases}
\]

Moreover,

\[
d^{(r+\ell)}_{kt - kr + r+1} \geq d^{(r+\ell)}_{kt - kr + r+1} - (r + \ell) \geq kt - kr - \ell - k \geq 2
\]

and the terms \(d^{(r+\ell)}_{kt - kr + r+1}, \ldots, d^{(r+\ell)}_{d_1 + 2}\) differ by at most one. This implies that \(p(\pi_{r+\ell}) = p(\pi)\).

Let \(\ell = \ell_1 + \ell_2 + \cdots + \ell_k\), where \(\ell_i = \lfloor \frac{\ell}{k} \rfloor + 1\) for each \(i = 1, \ldots, k\). In other words, \(\ell\) is partitioned into \(k\) parts of sizes \(\ell_1, \ldots, \ell_k\) as evenly as possible. Denote \(x_i = t - r - \ell_i\) for each \(i = 1, \ldots, k\). Then for each \(i = 1, \ldots, k\), we have

\[
kt - kr - \ell - 2 \geq d^{(r+\ell)}_{r+\ell+1} \geq \cdots \geq d^{(r+\ell)}_{kt - kr + r} \geq x_i \geq 1.
\]

Let

\[
da^{(r+\ell)}_{r+\ell+1} = f_{r+\ell+1} + (x_1 - 1) \text{ for } 1 \leq j \leq x_1,

d^{(r+\ell)}_{r+\ell+x_1, j} = f_{r+\ell+x_1, j} + (x_2 - 1) \text{ for } 1 \leq j \leq x_2,
\ldots
\]

\[
da^{(r+\ell)}_{r+\ell+x_1+\cdots+x_{k-1}, j} = f_{r+\ell+x_1+\cdots+x_{k-1}, j} + (x_k - 1) \text{ for } 1 \leq j \leq x_k.
\]
Clearly, $1 \leq f_{r+\ell+m} \leq (kt - kr - \ell - 2) - (t - r - \left\lfloor \frac{r}{k} \right\rfloor) + 2$ for each $m = 1, \ldots, kt - kr - \ell$. We now construct $\pi_i$, $r + \ell + 1 \leq i \leq kt - kr + r$ from $\pi_{i-1}$ by removing $d_i^{(i-1)}$, reducing the first $f_i$ nonzero terms, starting with $d_{kt-kr+r+1}^{(i-1)}$ by one, and then ordering the last $n - (kt - kr + r)$ terms to be non-increasing. Note that if $n$ is sufficiently large, then $p(\pi_{r+\ell}) = p(\pi) \geq \sqrt{\sigma(\pi)}$ (by Lemma 3.1) is also sufficiently large. Moreover, $f_{r+\ell+m} \leq kt - kr$ for each $m = 1, \ldots, kt - kr - \ell$. Thus, we can be assured that for $n$ large enough, there is a sufficient number of positive terms in each $\pi_{i-1}$ ($r + \ell + 1 \leq i \leq kt - kr + r$) to construct $\pi_i$ without forcing any terms in $\pi_i$ to be negative. We now present the following crucial lemma.

**Lemma 3.2** If $\pi_{kt-kr+r}$ is graphic, then $\pi$ is potentially $F_t,r,k$ (i.e. $K_r + kK_{t-r}$)-graphic.

**Proof.** Let $G_{kt-kr+r}$ be a realization of $\pi_{kt-kr+r}$ with $V(G_{kt-kr+r}) = \{v_{kt-kr+r+1}, \ldots, v_n\}$ and $d(v_{kt-kr+r+j}) = d_{kt-kr+r+j}^{(kt-kr+r)}$ for $1 \leq j \leq n - (kt - kr + r)$. Denote $\pi_0 = \pi$. The proof of Lemma 3.2 also depends on the following two cases.

**Case 1.** $d_{kt-kr+r-k-1} \geq kt - kr + r - 1$. For $i = kt - kr + r - 1, \ldots, 1, 0$ in turn, we can construct a realization $G_i$ of $\pi_i$ from the realization $G_{i+1}$ of $\pi_{i+1}$ by adding a new vertex $v_{i+1}$ to $G_{i+1}$ and joining it to the vertices whose degrees were reduced by one in going from $\pi_i$ to $\pi_{i+1}$.

Since $d_{kt-kr+r-k+1} \geq kt - kr + r - k$ (by Lemma 3.1), we have $d_{kt-kr+r-k}^{(kt-kr+r-k-1)} \geq d_{kt-kr+r-k+1}^{(kt-kr+r-k-1)} \geq 1$, and hence $v_{kt-kr+r-k}v_{kt-kr+r-k+1} \in E(G_{kt-kr+r-k-1})$. In creating $\pi_1, \ldots, \pi_{kt-kr+r-k-1}$, the fact that $d_{kt-kr+r-k-1} \geq kt - kr + r - 1$ implies that the realization $G_0$ of $\pi$ created in this manner will contain a copy of $K_r + (K_{kt-kr-k-1} + (K_2 \cup (k-1)K_1))$ such that $V(K_r) = \{v_1, \ldots, v_{\ell}\}$, $V(K_{kt-kr-k-1}) = \{v_{\ell+1}, \ldots, v_{kt-kr+r-k-1}\}$, $V(K_2) = \{v_{kt-kr+r-k}, v_{kt-kr+r-k+1}\}$ and $V((k-1)K_1) = \{v_{kt-kr+r-k+2}, \ldots, v_{kt-kr+r}\}$. It is easy to see that $K_{kt-kr-k-1} + (K_2 \cup (k-1)K_1)$ contains $kK_{t-r}$ as a subgraph. Thus, $G_0$ contains $K_r + kK_{t-r}$ as a subgraph.

**Case 2.** There is some $\ell$, $0 \leq \ell \leq kt - kr - k - 2$ such that $d_{r+\ell} \geq kt - kr - \ell - 1$ and $d_{r+\ell+1} \leq kt - kr + r - 2$. For $i = kt - kr + r - 1, \ldots, r + \ell + 1, r + \ell$ in turn, we can construct $G_i$ from $G_{i+1}$ by adding a new vertex $v_{i+1}$ to $G_{i+1}$ and joining it to vertices of those degrees that were reduced by one in the formation of $\pi_{i+1}$. It is easy to see that $G_{r+\ell}$ is a realization of $(f_{r+\ell+1}, \ldots, f_{kt-kr+r}, d_{kt-kr+r+1}^{(r+\ell)}, \ldots, d_n^{(r+\ell)})$ such that $d(v_{r+\ell+j}) = f_{r+\ell+j}$ for $1 \leq j \leq kt - kr - \ell$ and $\{v_{r+\ell+1}, \ldots, v_{kt-kr+r}\}$ forms an independent set in $G_{r+\ell}$.

We now construct a realization $G'_{r+\ell}$ of $\pi_{r+\ell}$ from $G_{r+\ell}$ by adding those edges such that
\{v_{r+\ell+x_0+\ldots+x_{j-1}+1}, \ldots, v_{r+\ell+x_0+\ldots+x_{j-1}+x_j}\} \) forms a clique for each \( j = 1, \ldots, k \), where \( x_0 = 0 \). For convenience, the graph \( G'_{r+\ell} \) is still denoted by \( G_{r+\ell} \). For \( i = r+\ell-1, \ldots, 1, 0 \) in turn, we then can construct a realization \( G_i \) of \( \pi_i \) from the realization \( G_{i+1} \) of \( \pi_{i+1} \) by adding a new vertex \( v_{i+1} \) to \( G_{i+1} \) and joining it to the vertices whose degrees were reduced by one in going from \( \pi_i \) to \( \pi_{i+1} \). The fact that \( d_{r+\ell} \geq kt - kr + r - 1 \) implies that \( G_0 \) contains \( K_r + (K_\ell + (K_{x_1} \cup \cdots \cup K_{x_k})) \) as a subgraph. It is easy to get that \( K_\ell + (K_{x_1} \cup \cdots \cup K_{x_k}) \) contains \( kK_\ell-r \) as a subgraph. Therefore, \( G_0 \) contains \( K_r + kK_\ell-r \) as a subgraph. \( \Box \)

**Proof of Theorem 1.1.** In order to prove

\[ \sigma(F_{t,r,k}, n) \leq (kt - kr - k + r - 1)(2n - kt + kr + k - r) + 2, \]

it is enough to prove that if \( \pi = (d_1, d_2, \ldots, d_n) \in GS_n \) with

\[ \sigma(\pi) \geq (kt - kr - k + r - 1)(2n - kt + kr + k - r) + 2, \]

then \( \pi \) is potentially \( F_{t,r,k} \) (i.e. \( K_r + kK_\ell-r \))-graphic. The proof follows by induction on \( r \). If \( r = 0 \), then \( \sigma(\pi) \geq (kt - k - 1)(2n - kt + k) + 2 \). By Theorem 2.6 (the case of \( m_1 = \cdots = m_k = t \)), \( \pi \) is potentially \( F_{t,0,k} \) (i.e. \( kK_t \))-graphic. Assume that the result holds for \( r - 1 \) (\( r \geq 1 \)). We will prove that the result holds for \( r \). Let \( \pi' = (d'_1, d'_2, \ldots, d'_{n-1}) \) be the residual sequence of \( \pi \). By Theorem 2.1, \( \pi' \) is graphic and

\[ \sigma(\pi') = \sigma(\pi) - 2d_1 \]
\[ \geq (kt - kr - k + r - 1)(2n - kt + kr + k - r) + 2 - 2(n - 1) \]
\[ = (k(t - 1) - k(r - 1) - k + (r - 1) - 1)(2(n - 1) - k(t - 1) + k(r - 1) + k - (r - 1)) + 2. \]

By the induction hypothesis, \( \pi' \) is potentially \( F_{t-1,r-1,k} \) (i.e. \( K_{r-1} + kK_{(t-1)-(r-1)} \))-graphic. In other words, there is some realization of \( \pi' \) that contains a copy of \( K_{r-1} + kK_{(t-1)-(r-1)} \). Furthermore, by Theorem 2.4, this implies that there exists a realization of \( \pi' \) with \( K_{r-1} + kK_{(t-1)-(r-1)} = K_{r-1} + kK_{t-r} \) on those vertices having degree \( d'_1, d'_2, \ldots, d'_{k(t-kr+r-1)} \). Now suppose that either \( d_1 = n - 1 \) or there exists an integer \( h, kt - kr + r \leq h \leq d_1 + 1 \) such that \( d_h > d_{h+1} \). Then \( d'_i = d_{i+1} - 1 \) for each \( i = 1, \ldots, kt - kr + r - 1 \). This implies that \( \pi \) would be potentially \( K_r + kK_{t-r} \)-graphic. Thus, we may assume that no such \( h \) exists and hence that

\[ n - 2 \geq d_1 \geq \cdots \geq d_{kt-kr+r} = \cdots = d_{d_1+2} \geq d_{d_1+3} \geq \cdots \geq d_n. \]
If $d_{2kt-2kr+2r} \geq kt - kr + r - 1$, then $\pi$ is potentially $K_{kt-kr+r}$-graphic by Theorem 2.5, which is sufficient to show that $\pi$ is potentially $K_r + kK_{t-r}$-graphic. We now may further assume that $d_{2kt-2kr+2r} \leq kt - kr + r - 2$. If $d_1 \leq 2kt - 2kr + 2r - 3$, then

$$
\sigma(\pi) \leq (2kt - 2kr + 2r - 3)(2kt - 2kr + 2r - 1) + (kt - kr + r - 2)(n - (2kt - 2kr + 2r - 1)).
$$

This is less than $(kt - kr - k + r - 1)(2n - kt + kr + k - r) + 2$ for $n$ sufficiently large. Hence $d_1 \geq 2kt - 2kr + 2r - 2$, i.e., $d_1 + 2 \geq 2kt - 2kr + 2r$. This implies that

$$
kt - kr + r - 2 \geq d_{kt-kr+r} = d_{kt-kr+r+1} = \cdots = d_{2kt-2kr+2r} = \cdots = d_{d_1+2}.
$$

For each $j = 1, \ldots, kt - kr + r$, the terms $d_{(kt-kr+r+1)}, \ldots, d_{(kt-kr+r+2)}, \ldots, d_{d_1+2}$ differ by at most one. Hence $\pi_{kt-kr+r}$ satisfies, for some $x \geq 1$,

$$
kt - kr + r - 2 \geq x = d_{(kt-kr+r+1)} \geq \cdots \geq d_{(kt-kr+2r+2)} \geq \cdots \geq d_{d_1+2} \geq x - 1.
$$

If $x = 1$, $\pi_{kt-kr+r}$ must be graphic as $\sigma(\pi_{kt-kr+r})$ is even. If $x \geq 2$, then

$$
\frac{1}{x-1} \left( \frac{(x + (x - 1) + 1)^2}{4} \right) \leq x + 2 \leq kt - kr + r.
$$

By Theorem 2.3, $\pi_{kt-kr+r}$ is also graphic. Thus, $\pi$ is potentially $K_r + kK_{t-r}$-graphic by Lemma 3.2. □

References


