STRUCTURED AND PUNCTURED NULLSTELLENSÄTZE

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ABSTRACT. A *Nullstellensatz* is a theorem providing information on polynomials that vanish on a certain set: David Hilbert's Nullstellensatz (1893) is a cornerstone of algebraic geometry, and Noga Alon's Combinatorial Nullstellensatz (1999) is a powerful tool in the *Polynomial Method*, a technique used in combinatorics. Alon's Theorem excludes that a polynomial vanishing on a grid contains a monomial with certain properties. This theorem has been generalized in several directions, two of which we will consider in detail: Terence Tao and Van H. Vu (2006), Uwe Schauz (2008) and Michał Lasoń (2010) exclude more monomials, and recently, Bogdan Nica (2023) improved the result for grids with additional symmetries in their side edges. Simeon Ball and Oriol Serra (2009) incorporated the multiplicity of zeros and gave Nullstellensätze for *punctured grids*, which are sets of the form $X \setminus Y$ with both X, Y grids.

We generalize some of these results; in particular, we provide a common generalization to the results of Schauz and Nica. To this end, we establish that during multivariate polynomial division, certain monomials are unaffected. This also allows us to generalize Pete L. Clark's proof of the nonzero counting theorem by Alon and Füredi to punctured grids.

1. INTRODUCTION

For a field $\mathbb{K}$, $n \in \mathbb{N}$ and a subset S of $\mathbb{K}^n$, we say that a polynomial $f \in \mathbb{K}[x_1, \dots, x_n]$ *vanishes on* S if $f(\boldsymbol{a}) = 0$ for all $\boldsymbol{a} \in S$. We will be particularly interested in the case that S is a grid. Here we say that a subset S of $\mathbb{K}^n$ is a *grid over* $\mathbb{K}$ if there are finite subsets $S_1, \dots, S_n$ of $\mathbb{K}$ such that $S = \bigtimes_{i=1}^n S_i$. For a polynomial $f \in \mathbb{K}[x_1, \dots, x_n]$, we denote its total degree by $\deg(f)$ (with $\deg(0) = -\infty$), and $\underline{n} := \{1, 2, \dots, n\}$. The model of our results will be Alon's Combinatorial Nullstellensatz.

Theorem 1.1 (Alon's Combinatorial Nullstellensatz [Alo99, Theorem 1.2]). *Let $S = \bigtimes_{i=1}^n S_i$ be a grid over $\mathbb{K}$, and let $f \in \mathbb{K}[x_1, \dots, x_n]$ be such that f contains a monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha_i < |S_i|$ for all $i \in \underline{n}$. Then if*

$$\sum_{i=1}^{n} \alpha_i = \deg(f), \tag{1.1}$$

there is $\boldsymbol{s} \in S$ such that $f(\boldsymbol{s}) \neq 0$.

The proof relies on the fact (see [Alo99, Theorem 1.1]) that the set $\mathbb{I}(S)$ of those polynomials in $\mathbb{K}[x_1, \dots, x_n]$ that vanish on S is the ideal of $\mathbb{K}[x_1, \dots, x_n]$ generated by $g_1, \dots, g_n$, where $g_i := \prod_{a \in S_i}(x_i - a)$, and on the fact that every $f \in \mathbb{I}(S)$ can be written as $\sum_{i=1}^n h_i g_i$ with

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$\deg(h_i g_i) \le \deg(f)$ for all $i \in \underline{n}$. For ensuring the existence of a nonzero on a grid, Alon's Theorem requires that f contains a monomial of maximal total degree such that the degree in each variable is smaller than the corresponding side length of the grid. Several subsequent results relax the condition (1.1) on such a monomial, and a simple condition was given in [Las10]: For $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$, we write $\alpha \sqsubseteq \beta$ if $\alpha_i \le \beta_i$ for all $i \in \underline{n}$, and $\alpha \sqsubset \beta$ if $\alpha \sqsubseteq \beta$ and $\alpha \ne \beta$. The monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is also written as $\boldsymbol{x}^\alpha$. Clearly, a monomial $\boldsymbol{x}^\alpha$ divides a monomial $\boldsymbol{x}^\beta$ if and only if $\alpha \sqsubseteq \beta$. For a polynomial $f = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha \boldsymbol{x}^\alpha$, we let $\mathrm{Mon}(f) := \{\boldsymbol{x}^\alpha \mid \alpha \in \mathbb{N}_0^n, c_\alpha \ne 0\}$ be the set of monomials that appear in f, and $\mathrm{Supp}(f) = \{\alpha \in \mathbb{N}_0^n \mid c_\alpha \ne 0\}$ be the set of exponents of these monomials, called the *support* of f. Now [Las10, Theorem 2] tells that Theorem 1.1 still holds if we replace (1.1) by the weaker condition

$$(\alpha_1, \dots, \alpha_n) \text{ is maximal in } \mathrm{Supp}(f) \text{ with respect to } \sqsubseteq . \tag{1.2}$$

Stated differently, (1.2) requires that $\mathrm{Supp}(f)$ does not contain a $\gamma \in \mathbb{N}_0^n$ with $\alpha \sqsubseteq \gamma$ and $\alpha \ne \gamma$. For fields of characteristic 0, this result had been contained in [TV06, Exercise 9.1.4]. This condition (1.2) can also be stated as

$$\text{for every monomial } \boldsymbol{x}^\gamma \in \mathrm{Mon}(f) \setminus \{\boldsymbol{x}^\alpha\}, \text{ there is } i \in \underline{n} \text{ such that } \gamma_i < \alpha_i. \tag{1.3}$$

A stronger result is given in [Sch08, Theorem 3.2(ii)]. This results tells that Theorem 1.1 still holds if we replace (1.1) by

$$\begin{aligned} &\text{for every monomial } \boldsymbol{x}^\gamma \in \mathrm{Mon}(f) \setminus \{\boldsymbol{x}^\alpha\}, \\ &\qquad\qquad \text{there is } i \in \underline{n} \text{ such that } \gamma_i \ne \alpha_i \text{ and } \gamma_i \le |S_i| - 1. \end{aligned} \tag{1.4}$$

Schauz's result also applies to rings other than fields. In the present note, we restrict our attention to grids over fields. In [BS09], Ball and Serra incorporate the multiplicity of zeros into Alon's theorem, and they extend the result from grids to *punctured grids*; these are sets that can be written as $X \setminus Y$ with both X and Y grids. Kós and Rónyai [KR12] generalized Alon's theorem to grids whose edges are *multisets*; such grids will be considered in Section 7.

Nica [Nic23, Theorem 3.1] gives a different lever to achieve generalizations of Theorem 1.1 by taking into account the structure of the grid. For $\lambda \in \mathbb{N}_0$, we call a univariate polynomial $f \in \mathbb{K}[x]$ of degree $\nu \in \mathbb{N}_0$ λ-*lacunary* if in f, all coefficients of x^α with $\nu - \lambda \le \alpha < \nu$ vanish. Then [Nic23] defines a finite set $A \subseteq \mathbb{K}$ to be λ-*null* if the polynomial $\prod_{a \in A}(x - a)$ is λ-lacunary. For example, over the complex numbers $\mathbb{C}$, the set $\{a \in \mathbb{C} \mid a^n = 1\}$ is $(n-1)$-null because the polynomial $x^n - 1$ is $(n-1)$-lacunary, every finite subset of a field $\mathbb{K}$ is 0-null, and a subset S of $\mathbb{K}$ is 1-null if $\sum_{a \in S} a = 0$. Nica's Theorem states:

Theorem 1.2 (Nica's Combinatorial Nullstellensatz for Structured Grids, [Nic23, Theorem 3.1])**.** *Let $S = \bigtimes_{i=1}^n S_i$ be a grid over the field $\mathbb{K}$, and let $\lambda \in \mathbb{N}_0$ be such that each S_i is λ-null. Let $f \in \mathbb{K}[x_1, \dots, x_n]$ and let $(\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ with $\alpha_i < |S_i|$ for all $i \in \underline{n}$ be such that f contains the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Then if*

$$\sum_{i=1}^n \alpha_i \ge \deg(f) - \lambda \tag{1.5}$$

there is $\boldsymbol{s} \in S$ *such that* $f(\boldsymbol{s}) \neq 0$.

We can therefore see that Alon's theorem has already been extended along four lines in the literature: In one line are the extensions of Ball and Serra [BS09] and Kós and Rónyai [KR12] including the *multiplicities of zeros*. Another line is the extension of the theorems to sets that are not grids as in Ball and Serra's extension to *punctured grids*. Nica's extension applies to *structured grids*, and Lasoń's, Tao and Vu's and Schauz's extensions put different conditions on the set of *monomials* appearing in the polynomial.

In the present paper, we combine these threads and obtain generalizations of some of these theorems. Essential in our proofs is an analysis of multivariate polynomial division; here we borrow some terms from the theory of Gröbner bases [Buc85, BW93, AL94].

2. Results

Our first result incorporates Nica's improvement of the Combinatorial Nullstellensatz for structured grids ([Nic23, Theorem 3.1]) into Schauz's result [Sch08, Theorem 3.2(ii)] and surprisingly yields more than the union of the two statements. For $a, b \in \mathbb{N}_0$, the interval $[a, b]$ is defined by $[a, b] := \{x \in \mathbb{N}_0 \mid a \leq x \leq b\}$. For $a > b$, we then have $[a, b] = \varnothing$.

Theorem 2.1 (Structured Nullstellensatz using conditions on the monomials)**.** *Let* $n \in \mathbb{N}$ *and let* $\lambda_1, \ldots, \lambda_n \in \mathbb{N}_0$. *For each* $i \in \underline{n}$, *let* S_i *be a* λ_i*-null subset of the field* $\mathbb{K}$, *and let* $S := \bigtimes_{i=1}^{n} S_i$. *Let* $f \in \mathbb{K}[x_1, \ldots, x_n]$ *and let* $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ *with* $\alpha_i < |S_i|$ *for all* $i \in \underline{n}$ *be such that* f *contains the monomial* $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. *Furthermore, we assume that for every monomial* $\boldsymbol{x}^\gamma$ *in* $\mathrm{Mon}(f) \setminus \{\boldsymbol{x}^\alpha\}$, *there is* $i \in \underline{n}$ *such that*

$$\gamma_i \in [0, \alpha_i - 1] \cup [\alpha_i + 1, |S_i| - 1] \cup [|S_i|, \alpha_i + \lambda_i]. \tag{2.1}$$

Then there is $\boldsymbol{s} \in S$ *such that* $f(\boldsymbol{s}) \neq 0$.

The proof is given in Section 5. We note that if we replace (2.1) by

$$\gamma_i \in [0, \alpha_i - 1],$$

we obtain Lasoń's result [Las10, Theorem 2], if we replace (2.1) by

$$\gamma_i \in [0, \alpha_i - 1] \cup [\alpha_i + 1, |S_i| - 1],$$

we obtain Schauz's result [Sch08, Theorem 3.2(ii)], and if we replace (2.1) by

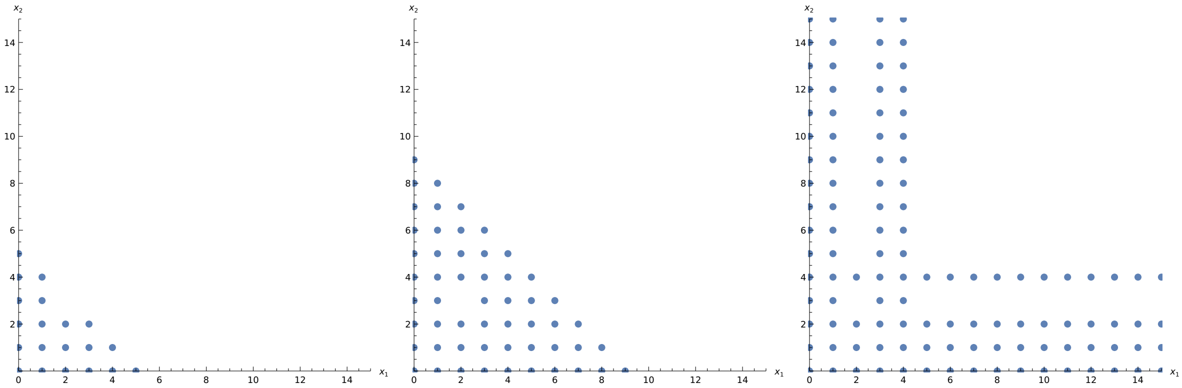
$$\gamma_i \in [0, \alpha_i - 1] \cup [\alpha_i + 1, \alpha_i + \lambda_i],$$

we obtain a result that implies Theorem 1.2. For this purpose, we note that

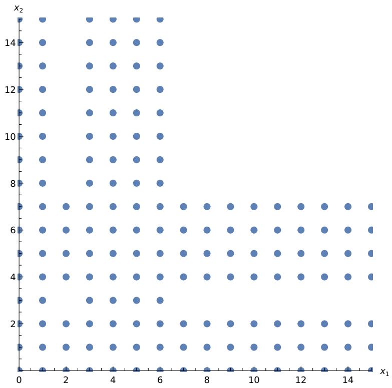
$$[\alpha_i + 1, \alpha_i + \lambda_i] \subseteq [\alpha_i + 1, |S_i| - 1] \cup [|S_i|, \alpha_i + \lambda_i]. \tag{2.2}$$

If $|S_i| - 1 > \alpha_i + \lambda_i$, then the inclusion is proper. Hence Theorem 2.1 generalizes these three results. An extension to multisets is given in Theorem 7.2.

Let us compare Theorem 2.1 to other Nullstellensätze by looking at an example: Consider the 4-null sets $S_1 = S_2 = \{z \in \mathbb{C} \mid z^5 = 1\}$, let $S := S_1 \times S_2$, and let $\boldsymbol{x}^\alpha := x_1^2 x_2^3$. Each of the



M_{A}, [Alo99, Theorem 1.2] M_{N}, [Nic23, Theorem 3.1] M_{S}, [Sch08, Theorem 3.2(ii)]



M_{ASZ}, Theorem 2.1

FIGURE 1. $x_1^2 x_2^3$ + any linear combinations of the dotted monomials does not vanish on $S = \{(x_1, x_2) \in \mathbb{C}^2 \mid x_1^5 = x_2^5 = 1\}$.

compared results yields a set of monomials M such that every polynomial that is a sum of $x_1^2 x_2^3$ and a linear combination of monomials in M has a nonzero in S. In Figure 1 (made with Mathematica [Wol24]) we draw the representations $\{(\gamma_1, \gamma_2) \in \mathbb{N}_0^2 \mid x_1^{\gamma_1} x_2^{\gamma_2} \in M\}$ of these sets of monomials. For an in-depth comparison of allowable monomials for various versions of the Nullstellensatz, we point the reader to [Rot23].

Our next goal is to incorporate multiplicities. Let $\mathbb{K}$ be a field, let $f \in \mathbb{K}[x_1, \ldots, x_n]$, let $t \in \mathbb{N}_0$, and let $\boldsymbol{c} = (c_1, \ldots, c_n) \in \mathbb{K}$. We say that $\boldsymbol{c}$ is a *zero of multiplicity* t or a t*-fold zero* of f if the polynomial $f' := f(c_1 + x_1, \ldots, c_n + x_n)$ lies in the ideal $\langle x_1, \ldots, x_n \rangle^t$, which holds if and only if f' contains no monomials of total degree less than t. We note that every $\boldsymbol{c} \in \mathbb{K}^n$ is a 0-fold zero of f, and that $\boldsymbol{c}$ is a 1-fold zero of f if and only if $f(\boldsymbol{c}) = 0$. Furthermore, in our definition, a t-fold zero is a t'-fold zero for all $t' \leq t$.

Theorem 2.2 (Structured Nullstellensatz using conditions on the monomials with multiplicities)**.** *Let* $n, t \in \mathbb{N}$ *and let* $\lambda \in \mathbb{N}_0^n$. *For each* $i \in \underline{n}$, *let* S_i *be a* λ_i*-null subset of the field* $\mathbb{K}$, *and let* $S := \bigtimes_{i=1}^n S_i$. *Let* $f \in \mathbb{K}[x_1, \ldots, x_n]$ *and let* $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ *be such that*

$$\text{for all } \beta \in \mathbb{N}_0^n : \sum_{i=1}^n \beta_i = t \Longrightarrow (\exists i \in \underline{n} : \alpha_i < \beta_i |S_i|) \tag{2.3}$$

and f contains the monomial $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. Furthermore, we assume that for every monomial $\boldsymbol{x}^\gamma$ in $\mathrm{Mon}(f)\setminus\{\boldsymbol{x}^\alpha\}$, there is $i\in \underline{n}$ with $\gamma_i\neq\alpha_i$ such that

$$\gamma_i \in [0,\alpha_i-1]\cup[\alpha_i+1,\alpha_i+\lambda_i] \text{ or } \forall\beta\in\mathbb{N}_0^n : \big((\sum_{j=1}^n \beta_j = t \text{ and } \beta_i>0) \Rightarrow (\exists j\in\underline{n} : \gamma_j<\beta_j|S_j|)\big). \tag{2.4}$$

Then there is $\boldsymbol{s}\in S$ such that $\boldsymbol{s}$ is not a t-fold zero of f.

The proof is given in Section 6. Setting $t=1$, we obtain Theorem 2.1 since $\boldsymbol{s}$ is a 1-fold zero of f if and only if $f(\boldsymbol{s})\neq 0$. As a corollary, we obtain a common generalization of [Nic23, Theorem 3.1] and [BS09, Corollary 3.2].

Corollary 2.3 (Structured Nullstellensatz with multiplicities)**.** *Let $n\in\mathbb{N}$ and let $\lambda\in\mathbb{N}_0$. For each $i\in\underline{n}$, let S_i be a λ-null subset of the field $\mathbb{K}$, and let $S:=\bigtimes_{i=1}^n S_i$. Let $f\in\mathbb{K}[x_1,\dots,x_n]$ and let $(\alpha_1,\dots,\alpha_n)\in\mathbb{N}_0^n$ be such that*

$$\text{for all } \beta\in\mathbb{N}_0^n : \sum_{i=1}^n \beta_i = t \Longrightarrow (\exists i\in\underline{n} : \alpha_i<\beta_i|S_i|),$$

f contains the monomial $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, and

$$\sum_{i=1}^n \alpha_i \geq \deg(f)-\lambda. \tag{2.5}$$

Then there is $\boldsymbol{s}\in S$ such that $\boldsymbol{s}$ is not a t-fold zero of f.

Next, we consider generalizations from grids to *punctured grids.*

Theorem 2.4 (Structured Nullstellensatz for punctured grids using conditions on the monomials)**.** *Let $X=\bigtimes_{i=1}^n X_i, Y=\bigtimes_{i=1}^n Y_i$ be grids over the field $\mathbb{K}$ with $Y_i\subsetneq X_i$ for all $i\in\underline{n}$, let $P:=X\setminus Y$, and let $\lambda_1,\dots,\lambda_n\in\mathbb{N}_0$. We assume that for each $i\in\underline{n}$, both X_i and Y_i are λ_i-null. Let $f\in\mathbb{K}[x_1,\dots,x_n]$ and let $(\alpha_1,\dots,\alpha_n)\in\mathrm{Supp}(f)$ be such that*

1. *for all $i\in\underline{n}$, $\alpha_i<|X_i|$,*
2. *there exists $i\in\underline{n}$ such that $\alpha_i<|X_i|-|Y_i|$,*
3. *for all $\boldsymbol{x}^\gamma\in\mathrm{Mon}(f)$, there exists $i\in\underline{n}$ such that at least one of the following three conditions holds:*
   - (a) *$\gamma_i\in[0,\alpha_i-1]\cup[\alpha_i+1,\alpha_i+\lambda_i]$,*
   - (b) *$\gamma_i\in[\alpha_i+1,|X_i|-1]$ and $|X_i|=|Y_i|$,*
   - (c) *$\gamma_i\in[\alpha_i+1,|X_i|-1]$ and there is $j\in\underline{n}$ with $\gamma_j<|X_j|-|Y_j|$.*

Then there is $\boldsymbol{z}\in P$ with $f(\boldsymbol{z})\neq 0$.

The proof is given in Section 8. As a consequence, we obtain:

Corollary 2.5 (Structured Nullstellensatz for punctured grids)**.** *Let $X=\bigtimes_{i=1}^n X_i, Y=\bigtimes_{i=1}^n Y_i$ be grids over the field $\mathbb{K}$ with $Y_i\subsetneq X_i$ for all $i\in\underline{n}$, let $P:=X\setminus Y$, and let $\lambda\in\mathbb{N}_0$. We assume that for each $i\in\underline{n}$, both X_i and Y_i are λ-null. Let $f\in\mathbb{K}[x_1,\dots,x_n]$ and let $(\alpha_1,\dots,\alpha_n)\in\mathrm{Supp}(f)$ be such that*

(1) *for all* $i \in \underline{n}$, $\alpha_i < |X_i|$,
(2) *there exists* $i \in \underline{n}$ *such that* $\alpha_i < |X_i| - |Y_i|$,
(3) $\sum_{i=1}^{n} \alpha_i \geq \deg(f) - \lambda$.

Then there is $\boldsymbol{z} \in P$ *with* $f(\boldsymbol{z}) \neq 0$.

The investigation of punctured grids also yields the following extension of the Alon-Füredi Nonzero Counting Theorem [AF93, Theorem 5]. The recent book manuscript by Clark [Cla24] contains other extensions and was the basis for ours. For $f \in \mathbb{K}[x_1, \ldots, x_n]$, we write $\mathbb{V}(f)$ for the set $\{\boldsymbol{a} \in \mathbb{K}^n \mid f(\boldsymbol{a}) = 0\}$ of zeros of f.

Theorem 2.6 (Nonzero counting for punctured grids)**.** *Let* $X = \bigtimes_{i=1}^{n} X_i$ *and* $Y = \bigtimes_{i=1}^{n} Y_i$ *be grids over the field* $\mathbb{K}$*, let* $P := X \setminus Y$*, and let* $f \in \mathbb{K}[x_1, \ldots, x_n] \setminus \{0\}$*. For* $i \in \underline{n}$*, let* $a_i := |X_i|$ *and* $b_i := |Y_i|$.

(1) *Let*

$$A := \{(y_1, \ldots, y_n) \in \mathbb{N}^n \mid \forall i \in \underline{n} : 1 \leq y_i \leq a_i,\ \exists i \in \underline{n} : y_i > b_i,\ \text{and} \sum_{i=1}^{n} y_i \geq \sum_{i=1}^{n} a_i - \deg(f)\}.$$

If $P \setminus \mathbb{V}(f) \neq \varnothing$*, then*

$$|P \setminus V(f)| \geq \min\{\prod_{i=1}^{n} y_i - \prod_{i=1}^{n} \min(y_i, b_i) \mid (y_1, \ldots, y_n) \in A\}. \tag{2.6}$$

(2) *We assume that for all* $i \in \underline{n}$*, we have* $\deg_{x_i}(f) < a_i$*. Let*

$$B := \{(y_1, \ldots, y_n) \in \mathbb{N}^n \mid \forall i \in \underline{n} : a_i - \deg_{x_i}(f) \leq y_i \leq a_i,\ \text{and} \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} a_i - \deg(f)\}.$$

If $P \setminus \mathbb{V}(f) \neq \varnothing$*, then*

$$|P \setminus V(f)| \geq \min\{\prod_{i=1}^{n} y_i - \prod_{i=1}^{n} \min(y_i, b_i) \mid (y_1, \ldots, y_n) \in B\}. \tag{2.7}$$

The proof is given in Section 9. Let us give an overview how these results are proved. Theorem 2.1 claims that a polynomial containing certain monomials does not vanish on the whole grid S. From [Alo99, Theorem 1.1] we know that the ideal $\mathbb{I}(S)$ of polynomials vanishing on S is generated by $\{g_i \mid i \in \underline{n}\}$ with $g_i := \prod_{a \in S_i}(x_i - a)$. The polynomial g_i has leading term $x_i^{|S_i|}$. For proving $f \notin \mathbb{I}(S)$, we show that its remainder r modulo $G = \{g_i \mid i \in \underline{n}\}$ after multivariate polynomial division by G is nonzero. The conditions on f and the g_i's ensure that the monomial $\boldsymbol{x}^\alpha$ can never be reduced in the course of the division, and that all other monomials in f have too small exponents to be able to produce a term $c_\alpha \boldsymbol{x}^\alpha$ that would allow to cancel $\boldsymbol{x}^\alpha$ before it stays – remains – in the remainder. Such an approach only works if all polynomials in $\mathbb{I}(S)$ will have remainder 0 after multivariate division by G. This is guaranteed when G is not only a generating set, but furthermore even a *Gröbner basis* of $\mathbb{I}(S)$; for the generating set G considered in the proof of Theorem 2.1 this is ensured by the fact that leading monomials of the polynomials in G are coprime to each other. In order to state these ideas precisely, we will make use of some notions from the arithmetic of multivariate polynomials, in particular of multivariate polynomial division, which is one of the basics of the theory of Gröbner bases.

3. Lacunary multivariate polynomials

In this section, we extend the definition of lacunary polynomials to multivariate polynomials.

We sort monomials using *admissible orderings:* A linear order $\leq_a$ on $\mathbb{N}_0^n$ is *admissible* if it is total, and for all $\alpha, \beta, \gamma \in \mathbb{N}_0^n$, we have $(\alpha \sqsubseteq \beta \Rightarrow \alpha \leq_a \beta)$ and $(\alpha \leq_a \beta \Rightarrow \alpha + \gamma \leq_a \beta + \gamma)$. When $\alpha \leq_a \beta$, we will also write $\boldsymbol{x}^\alpha \leq_a \boldsymbol{x}^\beta$, and $\alpha <_a \beta$ stands for ($\alpha \leq_a \beta$ and $\alpha \neq \beta$). If $\alpha \in \mathbb{N}_0^n$ is maximal in $\mathrm{Supp}(f)$ with respect to $\leq_a$, then $\boldsymbol{x}^\alpha$ is called the *leading monomial of* f and abbreviated by $\mathrm{LM}(f)$, and α is the *leading exponent* or *multidegree* of f, abbreviated by $\mathrm{LEXP}(f)$ and $\mathrm{mdeg}(f)$. The coefficient c_α of the leading monomial $\boldsymbol{x}^\alpha$ is the *leading coefficient*, abbreviated as $\mathrm{LC}(f)$, and $c_\alpha \boldsymbol{x}^\alpha = \mathrm{LC}(f) \cdot \mathrm{LM}(f) = \mathrm{LC}(f)\boldsymbol{x}^{\mathrm{LEXP}(f)}$ is the *leading term* of f, denoted by $\mathrm{LT}(f)$. Every admissible ordering is a well ordering, i.e., it is total and has no infinite descending chains (cf. [BW93, Theorem 5.5(ii)]; a proof can also be found, e.g., in the survey [Aic24] (Lemma 9.2)).

Definition 3.1. Let $\lambda \in \mathbb{N}_0^n$, and let $g \in \mathbb{K}[x_1, \ldots, x_n]$. The polynomial g is λ-*lacunary* if it contains a monomial $\boldsymbol{x}^\mu$ such that for each $\boldsymbol{x}^\nu \in \mathrm{Mon}(g)$ and for each $i \in \underline{n}$, we have $\nu_i < \mu_i - \lambda_i$ or $\nu_i = \mu_i$. A set $G \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is called λ-lacunary if every $g \in G$ is λ-lacunary.

We note that then $\nu \sqsubseteq \mu$ for all $\boldsymbol{x}^\nu \in \mathrm{Mon}(g)$, and therefore $\boldsymbol{x}^\mu$ is the leading monomial of g with respect to every admissible monomial ordering $\leq_a$.

Lemma 3.2. *Let $\lambda \in \mathbb{N}_0^n$, and let $f, g, h \in \mathbb{K}[x_1, \ldots, x_n]$ with $f = gh$. Then we have:*

(1) *If g and h are λ-lacunary, then f is λ-lacunary.*
(2) *If f and g are λ-lacunary, then h is λ-lacunary.*

Proof. Let $\boldsymbol{x}^\mu := \mathrm{LM}(f)$, $\boldsymbol{x}^\nu := \mathrm{LM}(g)$ and $\boldsymbol{x}^\rho := \mathrm{LM}(h)$. Then $\mu = \nu + \rho$. For showing (1), we fix $i \in \underline{n}$ and show that for each monomial $\boldsymbol{x}^\alpha \in \mathrm{Mon}(f)$, we have $\alpha_i = \mu_i$ or $\alpha_i < \mu_i - \lambda_i$. We write f as $\sum_{j=0}^{\mu_i} f_j x_i^j$, where $f_j \in \mathbb{K}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ for all j with $0 \leq j \leq \mu_i$, and we set $f_j = 0$ for $j > \mu_i$. Similarly,

$$g = \sum_{j=0}^{\nu_i} g_j x_i^j \text{ and } h = \sum_{j=0}^{\rho_i} h_j x_i^j. \tag{3.1}$$

Since g and h are λ-lacunary, we have $g_j = 0$ for all j with $\nu_i - \lambda_i \leq j \leq \nu_i - 1$ and $h_j = 0$ for all j with $\rho_i - \lambda_i \leq j \leq \rho_i - 1$. Now let $k \in \mathbb{N}$ be such that

$$\mu_i - \lambda_i \leq k \leq \mu_i - 1.$$

Then $f_k = \sum_{l=0}^{k} g_l h_{k-l}$. We will show $f_k = 0$ by establishing that all $k+1$ summands are 0. To this end, we let $l \in \{0, \ldots, k\}$. If $l < \nu_i - \lambda_i$, then $k - l > k - \nu_i + \lambda_i \geq \mu_i - \lambda_i - \nu_i + \lambda_i = \rho_i$, and thus $h_{k-l} = 0$. If $\nu_i - \lambda_i \leq l \leq \nu_i - 1$, we have $g_l = 0$. If $\nu_i \leq l \leq k - \rho_i + \lambda_i$, we have $k - l \geq \rho_i - \lambda_i$ and $k - l \leq k - \nu_i \leq \mu_i - 1 - \nu_i = \rho_i - 1$ and thus $h_{k-l} = 0$. If $l \geq k - \rho_i + \lambda_i + 1$, then $l \geq \mu_i - \lambda_i - \rho_i + \lambda_i + 1 = \nu_i + 1$, and therefore $g_l = 0$. Thus $f_k = 0$. Hence f contains no monomial $\boldsymbol{x}^\alpha$ with $\mu_i - \lambda_i \leq \alpha_i \leq \mu_i - 1$. This completes the proof of (1).

For proving (2), we assume that g is λ-lacunary and h is not λ-lacunary. Then there are $\boldsymbol{x}^\alpha \in \mathrm{Mon}(h)$ and $i \in \underline{n}$ with

$$\rho_i - \lambda_i \leq \alpha_i \leq \rho_i - 1.$$

Again, we write $f = \sum_{j=0}^{\mu_i} f_j x_i^j$, $g = \sum_{j=0}^{\nu_i} g_j x_i^j$ and $h = \sum_{j=0}^{\rho_i} h_j x_i^j$. Since $\boldsymbol{x}^\alpha \in \mathrm{Mon}(h)$, we have $h_{\alpha_i} \neq 0$. We have

$$f_{\nu_i+\alpha_i} = \sum_{l=0}^{\nu_i+\alpha_i} g_{\nu_i+\alpha_i-l} h_l.$$

For $l < \alpha_i$, we have $g_{\nu_i+\alpha_i-l} = 0$. For $l = \alpha_i$, we obtain the summand $g_{\nu_i} h_{\alpha_i}$. For l with $\alpha_i < l \leq \alpha_i + \lambda_i$, we obtain $\nu_i - \lambda_i \leq \nu_i + \alpha_i - l < \nu_i$, and therefore $g_{\nu_i+\alpha_i-l} h_l = 0$. For $l > \alpha_i + \lambda_i$, we have $l > \rho_i$, and therefore $h_l = 0$. Hence $f_{\nu_i+\alpha_i} = g_{\nu_i} h_{\alpha_i} \neq 0$. Since $\mu_i - \lambda_i = \nu_i + \rho_i - \lambda_i \leq \nu_i + \alpha_i \leq \nu_i + \rho_i - 1 = \mu_i - 1$, f can then not be λ-lacunary. □

4. Multivariate Polynomial Division

In this section, we analyze the stability of certain monomials during multivariate polynomial division. Over the integers, a division of f by g with $g \neq 0$ produces a quotient h and a remainder r with $f = hg + r$ and $|r| < |g|$. If $f, g_1, \ldots, g_s$ are multivariate polynomials in $\mathbb{K}[x_1, \ldots, x_n]$, then division produces an expression $f = \sum_{i=1}^{s} h_i g_i + r$ with certain properties of both the "quotients" $h_1, \ldots, h_s$ and the remainder r. Following [Eis95], the equation $f = \sum_{i=1}^{s} h_i g_i + r$ is then called a *standard expression*. We will need to write the h_i's as sums of monomials, and we observe that the multivariate polynomial division algorithm explained, e.g., in [BW93, Proposition 5.22], [CLO15, Chapter 2, §3], [AL94, Algorithm 1.5.1] or [Smi14, Algorithm 2.3.4] can easily be modified to produce what we will call a *natural standard expression*. Every natural standard expression with remainder 0 is a *standard representation* in the sense of [BW93, Definition 5.59].

Definition 4.1. Let $G \subseteq \mathbb{K}[x_1, \ldots, x_n] \setminus \{0\}$. A *natural standard expression* of f by G with remainder r with respect to the admissible ordering $\leq_a$ is an equality

$$f = \sum_{j=1}^{t} c_j \, \boldsymbol{x}^{\delta_j} g_j + r, \tag{4.1}$$

where $t \in \mathbb{N}_0$, $c_1, \ldots, c_t \in \mathbb{K} \setminus \{0\}$, $\delta_1, \ldots, \delta_t \in \mathbb{N}_0^n$, $g_1, \ldots, g_t \in G$, $r \in \mathbb{K}[x_1, \ldots, x_n]$, and for each $j \in \underline{t}$, we have

$$\mathrm{LM}(c_j \, \boldsymbol{x}^{\delta_j} g_j) \in \mathrm{Mon}(f - \sum_{i=1}^{j-1} c_i \boldsymbol{x}^{\delta_i} g_i) \tag{4.2}$$

and

$$\mathrm{LM}(c_j \, \boldsymbol{x}^{\delta_j} g_j) \notin \mathrm{Mon}(f - \sum_{i=1}^{j} c_i \boldsymbol{x}^{\delta_i} g_i), \tag{4.3}$$

and r does not contain a monomial that is divisible by any monomial in $\{\mathrm{LM}(g) \mid g \in G\}$.

This definition expresses that during the j-th step in the division of f by G, the term $c_j \boldsymbol{x}^{\delta_j} \mathrm{LM}(g_j)$ appears in the intermediate polynomial that we seek to reduce, and this term is eliminated by subtracting $c_j \boldsymbol{x}^{\delta_j} g_j$. There are two differences to standard expressions as used in the literature: first (cf. [Eis95, p.334]), standard expressions are often written in a collected form $\sum_{i=1}^{s} h_i g_i + r$ with $h_i \in \mathbb{K}[x_1, \ldots, x_n]$. The second main difference is that a standard representation as defined in [BW93, Definition 5.59] need not come from an actual execution of the division algorithm; for example setting $f = 2x^2y + x$, $g_1 = xy$, $g_2 = x^2$, we obtain $f = x \cdot g_1 + y \cdot g_2 + x$, which is a standard representation, but an execution of the division algorithm would always reduce the monomial $2x^2y$ in one step, yielding, e.g., the natural standard expression $f = 2xg_1 + x$. The definitions in [Eis95, BW93] do not grasp this aspect of division, and hence for our purposes, we prefer the refinement to *natural standard expressions* given in Definition 4.1.

An important observation is that during this division process, certain monomials of f can never be reduced and will therefore end up in the remainder r. We will always assume that the divisors are λ-lacunary polynomials. We note that for a λ-lacunary polynomial g, the leading monomial is the same for all admissible monomial orderings $\leq_a$. Hence the following definitions do not depend on the choice of the admissible monomial ordering $\leq_a$ used to determine $\mathrm{LM}(g)$. The first definition tries to identify monomials $\boldsymbol{x}^\gamma$ in a polynomial f that, in the course of a multivariate polynomial division of f by G, have the potential to produce a term $c\boldsymbol{x}^\alpha$ that might cancel $\boldsymbol{x}^\alpha$. We will call these threats to $\boldsymbol{x}^\alpha$'s ability to remain intact during the division process (G, λ, α)-*shading monomials.*

Definition 4.2. Let $\alpha, \gamma, \lambda \in \mathbb{N}_0^n$ and let G be a λ-lacunary subset of $\mathbb{K}[x_1, \ldots, x_n]$. The monomial $\boldsymbol{x}^\gamma$ is (G, λ, α)-*shading* if

(1) $\alpha \sqsubseteq \gamma$ and $\alpha \neq \gamma$,
(2) for all $i \in \underline{n}$ with $\alpha_i < \gamma_i$, there is $g \in G$ with $\mathrm{LM}(g) \mid \boldsymbol{x}^\gamma$ and $\deg_{x_i}(g) > 0$, and
(3) for all $i \in \underline{n}$ with $\alpha_i < \gamma_i$, we have $\alpha_i + \lambda_i < \gamma_i$.

The next definition tries to single out monomials that will not be affected by division by G. Theorem 4.4 then shows that these monomials indeed remain intact.

Definition 4.3. Let $G \subseteq \mathbb{K}[x_1, \ldots, x_n]$, let $\alpha, \lambda \in \mathbb{N}_0^n$, and let $f \in \mathbb{K}[x_1, \ldots, x_n]$. We say that $\boldsymbol{x}^\alpha$ is a (G, λ)-*stable* monomial in f if the following conditions hold:

(1) $\alpha \in \mathrm{Supp}(f)$,
(2) there is no $g \in G$ with $\mathrm{LM}(g) \mid \boldsymbol{x}^\alpha$, and
(3) f contains no (G, λ, α)-shading monomial.

Theorem 4.4. *Let $\lambda \in \mathbb{N}_0^n$, and let G be a λ-lacunary subset of $\mathbb{K}[x_1, \ldots, x_n]$. Let $f \in \mathbb{K}[x_1, \ldots, x_n]$, and let $g \in G$, $\delta \in \mathbb{N}_0^n$ be such that $\mathrm{LM}(\boldsymbol{x}^\delta g) \in \mathrm{Mon}(f)$. Let $\boldsymbol{x}^\alpha$ be a (G, λ)-stable monomial in f, let $c \in \mathbb{K} \setminus \{0\}$, and let*

$$h = f - c\,\boldsymbol{x}^\delta g.$$

Then $\boldsymbol{x}^\alpha$ is a (G, λ)-stable monomial in h.

Proof. Let $\mu := \textsc{Lexp}(g)$. We first show Condition (1) of Definition 4.3. This condition is

$$\alpha \in \mathrm{Supp}(h). \tag{4.4}$$

Seeking a contradiction, we suppose $\alpha \notin \mathrm{Supp}(h)$. Then $\alpha \in \mathrm{Supp}(\boldsymbol{x}^\delta g)$. Thus there is $\boldsymbol{x}^\varepsilon \in \mathrm{Mon}(g)$ with $\boldsymbol{x}^\alpha = \boldsymbol{x}^\delta \boldsymbol{x}^\varepsilon$. If $\boldsymbol{x}^\varepsilon = \textsc{Lm}(g)$, then $\textsc{Lm}(g) \mid \boldsymbol{x}^\alpha$, which violates Condition (2) of Definition 4.3, and hence $\boldsymbol{x}^\alpha$ is not (G, λ)-stable in f, contradicting the assumptions. In the case that $\boldsymbol{x}^\varepsilon \neq \textsc{Lm}(g)$, we show that

$$\boldsymbol{x}^\delta \boldsymbol{x}^\mu \text{ is a } (G, \lambda, \alpha)\text{-shading monomial in } f. \tag{4.5}$$

To this end, we first show that $\boldsymbol{x}^\delta \boldsymbol{x}^\mu$ satisfies Condition (1) of Definition 4.2. Since g is lacunary, we have $\varepsilon \sqsubseteq \mu$, and therefore $\alpha = \delta + \varepsilon \sqsubseteq \delta + \mu$. Since $\varepsilon \neq \mu$, we also have $\alpha \neq \delta + \mu$. This completes the proof of Condition (1) of Definition 4.2. Next, we show Conditions (2) and (3) of Definition 4.2. To this end, we fix $i \in \underline{n}$ and assume $\alpha_i < \delta_i + \mu_i$. Then $\delta_i + \varepsilon_i < \delta_i + \mu_i$, and therefore $\varepsilon_i < \mu_i$. Hence $\mu_i \neq 0$, and thus g witnesses that Condition (2) is satisfied. Since G is λ-lacunary, we obtain $\varepsilon_i < \mu_i - \lambda_i$, which implies $\alpha_i + \lambda_i = \delta_i + \varepsilon_i + \lambda_i < \delta_i + \mu_i$, completing the proof of Condition (3) and of (4.5). Since $\boldsymbol{x}^\delta \boldsymbol{x}^\mu \in \mathrm{Mon}(f)$, this monomial violates Condition (3) of Definition 4.3 and therefore witnesses that $\boldsymbol{x}^\alpha$ is not (G, λ)-stable in f, contradicting the assumptions and completing the proof of (4.4).

Continuing to show that $\boldsymbol{x}^\alpha$ is (G, λ)-stable in h, we observe that Condition (2) of Definition 4.3 is inherited from the assumption that $\boldsymbol{x}^\alpha$ is (G, λ)-stable in f. Hence we turn to Condition (3). Seeking a contradiction, we assume that h contains a (G, λ, α)-shading monomial $\boldsymbol{x}^\gamma$. Since f contains no (G, λ, α)-shading monomial, we know that $\boldsymbol{x}^\gamma \in \mathrm{Mon}(\boldsymbol{x}^\delta g)$ and $\boldsymbol{x}^\gamma \neq \boldsymbol{x}^\delta \boldsymbol{x}^\mu$. Thus there is $\rho \in \mathrm{Supp}(g) \setminus \{\mu\}$ such that

$$\boldsymbol{x}^\gamma = \boldsymbol{x}^\delta \boldsymbol{x}^\rho.$$

Let $\tilde{\gamma} := \delta + \mu$. We show that then

$$\boldsymbol{x}^{\tilde{\gamma}} \text{ is a } (G, \lambda, \alpha)\text{-shading monomial in } f. \tag{4.6}$$

To this end, we first show that $\boldsymbol{x}^{\tilde{\gamma}}$ satisfies Condition (1) of Definition 4.2. Since g is lacunary, we have $\rho \sqsubseteq \mu$, and therefore

$$\gamma = \delta + \rho \sqsubseteq \delta + \mu = \tilde{\gamma}. \tag{4.7}$$

Since $\boldsymbol{x}^\gamma$ is (G, λ, α)-shading, we have $\alpha \sqsubset \gamma$, and therefore $\alpha \sqsubset \tilde{\gamma}$. This completes the proof of Condition (1) of Definition 4.2. Next, we show Conditions (2) and (3) of Definition 4.2. To this end, we fix $i \in \underline{n}$ and assume that $\alpha_i < \tilde{\gamma}_i$.

We first consider the case that $\alpha_i < \gamma_i$. Since $\boldsymbol{x}^\gamma$ is (G, λ, α)-shading in h, we obtain a $g' \in G$ with $\deg_{x_i}(g') > 0$ and $\textsc{Lm}(g') \mid \boldsymbol{x}^\gamma$, and that $\alpha_i + \lambda_i < \gamma_i$. Since $\gamma \sqsubseteq \tilde{\gamma}$, we then have $\textsc{Lm}(g') \mid \boldsymbol{x}^{\tilde{\gamma}}$ and $\alpha_i + \lambda_i < \tilde{\gamma}_i$. Thus in this case, (4.6) holds.

Now consider the case $\alpha_i = \gamma_i$. Then $\gamma_i < \tilde{\gamma}_i$, and thus $\rho_i < \mu_i$. We claim that then g witnesses Condition (2) of Definition 4.2 needed to verify for proving that $\boldsymbol{x}^{\tilde{\gamma}}$ is (G, λ, α)-shading. By the definition of $\tilde{\gamma}$, we obtain $\textsc{Lm}(g) = \boldsymbol{x}^\mu \mid \boldsymbol{x}^\delta \boldsymbol{x}^\mu = \boldsymbol{x}^{\tilde{\gamma}}$. Since $\mu_i > \rho_i$, we have $\deg_{x_i}(g) > 0$. We still have to show Condition (3), which claims that

$$\alpha_i + \lambda_i < \tilde{\gamma}_i. \tag{4.8}$$

Since $\rho_i < \mu_i$, the fact that g is λ-lacunary yields $\rho_i < \mu_i - \lambda_i$, and therefore $\alpha_i = \gamma_i = \delta_i + \rho_i < \delta_i + \mu_i - \lambda_i = \tilde{\gamma}_i - \lambda_i$, establishing (4.8) and completing the proof of (4.6). Since $\boldsymbol{x}^\delta \boldsymbol{x}^\mu \in \operatorname{Mon}(f)$, this monomial violates Condition (3) of Definition 4.3 and therefore witnesses that $\boldsymbol{x}^\alpha$ is not (G, λ)-stable in f, contradicting the assumptions and completing the proof that $\boldsymbol{x}^\alpha$ is (G, λ)-stable in h. □

Corollary 4.5. *Let $\lambda \in \mathbb{N}_0^n$, let G be a λ-lacunary subset of $\mathbb{K}[x_1, \ldots, x_n]$, and let $\leq_a$ be an admissible ordering of $\mathbb{N}_0^n$. Let*

$$f = \sum_{j=1}^{t} c_j \, \boldsymbol{x}^{\delta_j} g_j + r \tag{4.9}$$

be a natural standard expression of f by G. Then all (G, λ)-stable elements of $\operatorname{Mon}(f)$ *are also elements of* $\operatorname{Mon}(r)$*.*

Proof. Let $\boldsymbol{x}^\alpha$ be a (G, λ)-stable monomial in f. By induction on s, we show that $\boldsymbol{x}^\alpha$ is also (G, λ)-stable in

$$f - \sum_{j=1}^{s} c_j \, \boldsymbol{x}^{\delta_j} g_j.$$

For $s = 0$, there is nothing to prove. Now assume that $s \in \{0, \ldots, t-1\}$. As inductive hypothesis, we assume that $\boldsymbol{x}^\alpha$ is (G, λ)-stable in $f - \sum_{j=1}^{s} c_j \, \boldsymbol{x}^{\delta_j} g_j$. Then by Theorem 4.4, $\boldsymbol{x}^\alpha$ is also (G, λ)-stable in $f - \sum_{j=1}^{s} c_j \, \boldsymbol{x}^{\delta_j} g_j - c_{s+1} \boldsymbol{x}^{\delta_{s+1}} g_{s+1}$, completing the induction step. □

5. A Nullstellensatz for structured grids using conditions on the monomials

In this Section, we will prove Theorem 2.1. For $i \in \underline{n}$, we let $f_i(x) := \prod_{a \in S_i} (x - a)$, and we let $g_i := f_i(x_i)$. If each f_i is a univariate λ_i-lacunary polynomial in $\mathbb{K}[x]$, then for each $i \in \underline{n}$, g_i is a $(\lambda_1, \ldots, \lambda_n)$-lacunary polynomial in $\mathbb{K}[x_1, \ldots, x_n]$. With this observation in mind, we can apply Corollary 4.5 to prove the main result:

Proof of Theorem 2.1. Let $\lambda := (\lambda_1, \ldots, \lambda_n)$, and for each $i \in \underline{n}$, let $g_i := \prod_{a \in S_i} (x_i - a)$. Let I be the ideal of $\mathbb{K}[x_1, \ldots, x_n]$ generated by $G = \{g_1, \ldots, g_n\}$. By [Alo99, Theorem 1.1], a polynomial f vanishes on $S_1 \times \cdots \times S_n$ if and only if it lies in I. Now we seek to apply Corollary 4.5. For each $i \in \underline{n}$, S_i is λ_i-null and therefore the polynomial g_i is $(\lambda_1, \ldots, \lambda_n)$-lacunary. We will now show that $\boldsymbol{x}^\alpha$ is a (G, λ)-stable monomial in f with respect to $\leq_a$. First, we observe that for each $i \in \underline{n}$ we have $\alpha_i < |S_i|$ and therefore the monomial $\operatorname{LM}(g_i) = x_i^{|S_i|}$ does not divide $\boldsymbol{x}^\alpha$, which establishes Condition (2) of Definition 4.3. Next, we show that f contains no (G, λ, α)-shading monomial. Let $\boldsymbol{x}^\gamma \in \operatorname{Mon}(f)$. If $\gamma = \alpha$, then $\boldsymbol{x}^\gamma$ violates Condition (1) of Definition 4.2 and is therefore not (G, λ, α)-shading. If $\gamma \neq \alpha$, the assumption yields an $i \in \underline{n}$ such that

$$\gamma_i \in [0, \alpha_i - 1] \cup [\alpha_i + 1, |S_i| - 1] \cup [|S_i|, \alpha_i + \lambda_i].$$

If $\gamma_i \in [0, \alpha_i - 1]$, then Condition (1) of Definition 4.2 is violated, and so $\boldsymbol{x}^\gamma$ is not (G, λ, α)-shading. If $\gamma_i \in [\alpha_i + 1, |S_i| - 1]$ and Condition (2) of Definition 4.2 is satisfied, then $\operatorname{LM}(g_i) \mid \boldsymbol{x}^\gamma$, and therefore $|S_i| \leq \gamma_i$, contradicting $\gamma_i \leq |S_i| - 1$. We conclude that also in the case $\gamma_i \in [\alpha_i + 1, |S_i| - 1]$, $\boldsymbol{x}^\gamma$ is not (G, λ, α)-shading. If $\gamma_i \in [|S_i|, \alpha_i + \lambda_i]$, then we have $\gamma_i \geq |S_i| > \alpha_i$. If

Condition (3) of Definition 4.2 is satisfied, we have $\gamma_i > \alpha_i+\lambda_i$, contradicting $\gamma_i \leq \alpha_i+\lambda_i$. Hence also in this case $\boldsymbol{x}^\gamma$ is not (G,λ,α)-shading. Thus f contains no (G,λ,α)-shading monomial and therefore, $\boldsymbol{x}^\alpha$ is (G,λ)-stable.

Let $f = \sum_{j=1}^{t} c_j\, \boldsymbol{x}^{\delta_j} g_{i_j} + r$ be a natural standard expression of f by G. Since $\boldsymbol{x}^\alpha$ is a (G,λ)-stable monomial in f, Corollary 4.5 yields that $\boldsymbol{x}^\alpha \in \mathrm{Mon}(r)$. Since the leading monomials of the polynomials in G are coprime, [BW93, Lemma 5.66] yields that the set G is a Gröbner basis of I (cf. [CLO15, p.89, Exercise 11]). Since then all elements of I have zero remainder in every standard expression by G, we obtain $f \notin I$, and therefore, f does not vanish on all points in $S_1 \times \cdots \times S_n$. □

6. A Nullstellensatz for structured grids using conditions on the monomials with multiplicity

In this section, we will prove Theorem 2.2 and Corollary 2.3. Let $\mathbb{K}$ be a field, and let $t \in \mathbb{N}$. We define $\mathbb{I}_t(X)$ as the set of all $f \in \mathbb{K}[x_1,\ldots,x_n]$ that have a t-fold zero at each $\boldsymbol{a} \in X$. Hence $\mathbb{I}_0(X) = \mathbb{K}[x_1,\ldots,x_n]$ since every place is a 0-fold zero of every polynomial, and $\mathbb{I}_1(X) = \mathbb{I}(X)$. For X being a grid, [BS09] provides a basis of $\mathbb{I}_t(X)$. For arbitrary finite X, generators of $\mathbb{I}_t(X)$ can be determined from generators of $\mathbb{I}(X)$ using some arguments on ideals in commutative rings that we collect in the next two lemmata. For an ideal I of $\mathbb{K}[x_1,\ldots,x_n]$ and $t \in \mathbb{N}$, I^t denotes the t-th power of the ideal I, which is defined to be the ideal generated by all products $i_1 \cdots i_t$ with (not necessarily distinct) $i_1,\ldots,i_t \in I$. The product of two ideals I, J is the ideal generated by $\{ij \mid i \in I, j \in J\}$ and denoted by IJ.

Lemma 6.1. *Let R be a commutative ring with unit, let $s,t \in \mathbb{N}$, and let $M_1,\ldots,M_s$ be distinct maximal ideals of R. Then $(\bigcap_{i\in\underline{s}} M_i)^t = \bigcap_{i\in\underline{s}} M_i^t$.*

Proof. We proceed by induction on s. For $s=1$, the statement is obvious. For the induction step, let $s \geq 2$, and let $J := \bigcap_{i=2}^{s} M_i$. First, we observe that for any collection $I_1,\ldots,I_k$ of ideals with $I_i \not\subseteq M_1$ for all $i \in \underline{k}$, we have

$$I_1 I_2 \cdots I_k \not\subseteq M_1. \tag{6.1}$$

In order to show (6.1), we observe that for $i \in \underline{s}$ with $i \geq 2$, the assumption $I_i \not\subseteq M_1$ implies that there is $a_i \in I_i \setminus M_1$. The ideal M_1 is maximal, and therefore prime, and thus $\prod_{i=1}^{k} a_i \notin M_1$ and $\prod_{i=1}^{k} a_i \in I_1 \cdots I_k$. This proves (6.1). Setting $k := s-1$ and $I_j := M_{j+1}$, we obtain $M_2 \cdots M_s \not\subseteq M_1$, and since $M_2 \cdots M_s \subseteq J$ also

$$J \not\subseteq M_1. \tag{6.2}$$

Next, we show that for all ideals I of R with $I \not\subseteq M_1$ and for all $r \in \mathbb{N}$, we have

$$I \cap M_1^r = I M_1^r. \tag{6.3}$$

The $\supseteq$-inclusion is obvious, so we only prove $\subseteq$. Let $g \in I \cap M_1^r$. Since $I \not\subseteq M_1$, we have $I + M_1 = R$ and thus there are $a \in I$ and $b \in M_1$ such that $1 = a+b$. Then $g = (a+b)^r g = b^r g + \sum_{i=1}^{r} \binom{r}{i} a^i b^{r-i} g$. Then $b^r g \in M_1^r I = I M_1^r$, and for $i \geq 1$, we have $a^i \in I$ and $g \in M_1^r$, and therefore $a^i b^{r-i} g \in I M_1^r$. Thus $g \in I M_1^r$, which establishes (6.3). Now $\bigcap_{i\in\underline{s}} M_i^t = M_1^t \cap$

$\bigcap_{i=2}^{s} M_i^t$. By the induction hypothesis, the last expression is equal to $M_1^t \cap (\bigcap_{i=2}^{s} M_i)^t = M_1^t \cap J^t$. By (6.2) and (6.1), we have $J^t \not\subseteq M_1$, and thus by (6.3), we have $M_1^t \cap J^t = M_1^t J^t = (M_1 J)^t$. By (6.2) and (6.3), $(M_1 J)^t = (M_1 \cap J)^t = (\bigcap_{i=1}^{s} M_i)^t$. □

Lemma 6.2. *Let X be a finite subset of $\mathbb{K}^n$, and let $t \in \mathbb{N}$. Then $\mathbb{I}_t(X) = \mathbb{I}(X)^t$.*

Proof. For $\boldsymbol{a} \in X$, let $M_{\boldsymbol{a}} := \mathbb{I}(\{\boldsymbol{a}\})$. Then $\mathbb{I}(X) = \bigcap_{\boldsymbol{a} \in X} M_{\boldsymbol{a}}$. We observe that by definition, $\boldsymbol{a}$ is a t-fold zero of f if and only if $f(\boldsymbol{x} + \boldsymbol{a})$ lies in the ideal $\langle x_1, \dots, x_n \rangle^t$. Applying the isomorphism $\sigma : \mathbb{K}[x_1, \dots, x_n] \to \mathbb{K}[x_1, \dots, x_n]$, with $\sigma(x_i) := x_i - a_i$, we obtain that $f(\boldsymbol{x} + \boldsymbol{a}) \in \langle x_1, \dots, x_n \rangle^t$ if and only if $f(\boldsymbol{x}) \in \langle x_1 - a_1, \dots, x_1 - a_n \rangle^t$, which is equivalent to $f(\boldsymbol{x}) \in M_{\boldsymbol{a}}^t$. Hence $\mathbb{I}_t(X) = \bigcap_{\boldsymbol{a} \in X} M_{\boldsymbol{a}}^t$. Now Lemma 6.1 yields the required equality $\mathbb{I}(X)^t = \mathbb{I}_t(X)$. □

As a consequence, we obtain a set of generators of $\mathbb{I}_t(S)$ for a grid S. For a finite subset G of $\mathbb{K}[x_1, \dots, x_n]$, we define $G^t := \{g_1 \cdots g_t \mid g_1, \dots, g_t \in G\}$. The ideal of $\mathbb{K}[x_1, \dots, x_n]$ that is generated by G is denoted by $\langle G \rangle$.

Lemma 6.3 ([BS09, Theorem 3.1])**.** *Let $S = \bigtimes_{i=1}^{n} S_i$ be a grid over $\mathbb{K}$, for each $i \in \underline{n}$, let $g_i := \prod_{a \in S_i} (x_i - a)$, and let $G := \{g_1, \dots, g_n\}$ Then G^t generates the ideal $\mathbb{I}_t(S)$.*

Proof. By [Alo99, Theorem 1.1], G generates the ideal $\mathbb{I}(S)$. Therefore, G^t generates the ideal $\mathbb{I}(S)^t$, which is equal to $\mathbb{I}_t(S)$ by Lemma 6.2. □

Lemma 6.3 yields that every $f \in \mathbb{I}_t(S)$ can be written as $f = \sum_{i=1}^{k} h_i g_i'$ with $k \in N_0$, $h_1, \dots, h_k \in \mathbb{K}[x_1, \dots, x_n]$ and $g_1', \dots, g_k' \in G^t$. As an additional piece of information, [BS09, Theorem 3.1] ensures that we can pick these summands in a way that $\deg(h_k g_k') \le \deg(f)$. We give an alternative argument for these degree bounds by showing that G^t is a Gröbner basis. For this purpose, we extend Buchberger's First Criterion [BW93, Lemma 5.66].

Theorem 6.4. *Let $s, t \in \mathbb{N}$, and let $g_1, \dots, g_s \in \mathbb{K}[x_1, \dots, x_n] \setminus \{0\}$ be such that for $i, j \in \underline{s}$ with $i \neq j$, $\textsc{Lm}(g_i)$ and $\textsc{Lm}(g_j)$ do not have any variable in common, i.e., $\gcd(\textsc{Lm}(g_i), \textsc{Lm}(g_j)) = 1$, and let $\le_a$ be an admissible ordering of monomials. Then $G^t := \{g_1^{\alpha_1} \cdots g_s^{\alpha_s} \mid \alpha_1, \dots, \alpha_s \in \mathbb{N}_0, \sum_{i=1}^{s} \alpha_i = t\}$ is a Gröbner basis of the ideal $\langle G \rangle^t$ with respect to $\le_a$.*

Proof. Clearly, the set G^t generates the ideal $\langle G \rangle^t$. We now show that G^t is a Gröbner basis. To this end, we use Buchberger's Characterization Theorem for Gröbner bases [BW93, Theorem 5.64] (cf. [Buc85, Theorem 6.2], [Eis95, Theorem 15.8]) which states that G^t is a Gröbner basis if for all $f, h \in G^t$, the S-polynomial $S(f, h)$ has a standard expression by G^t with remainder 0. Without loss of generality, we assume that $g_1, \dots, g_s$ are normed, i.e., $\textsc{Lc}(g_i) = 1$ for all $i \in \underline{s}$, and we show that every S-polynomial of G^t has a standard expression by G^t with remainder 0. To this end, let $f = g_1^{\alpha_1} \cdots g_s^{\alpha_s}$ and $h = g_1^{\beta_1} \cdots g_s^{\beta_s}$ be elements of G^t. The S-polynomial $S(f, h)$ can be computed as

$$S(f, h) = \frac{\operatorname{lcm}(\textsc{Lm}(f), \textsc{Lm}(h))}{\textsc{Lm}(f)} f - \frac{\operatorname{lcm}(\textsc{Lm}(f), \textsc{Lm}(h))}{\textsc{Lm}(h)} h.$$

We have

$$\begin{aligned}\operatorname{lcm}(\mathrm{LM}(f),\mathrm{LM}(h)) &= \operatorname{lcm}\big(\mathrm{LM}(\prod_{i=1}^{s} g_i^{\alpha_i}),\mathrm{LM}(\prod_{i=1}^{s} g_i^{\beta_i})\big)\\ &= \operatorname{lcm}\big(\prod_{i=1}^{s}\mathrm{LM}(g_i)^{\alpha_i},\prod_{i=1}^{s}\mathrm{LM}(g_i)^{\beta_i}\big)\\ &= \prod_{i=1}^{s}\mathrm{LM}(g_i)^{\max(\alpha_i,\beta_i)},\end{aligned}$$

where the last equality holds because the $\mathrm{LM}(g_i)$ are coprime. Now let $I=\{i\in\underline{s}\mid \alpha_i>\beta_i\}$ and $J=\{j\in\underline{s}\mid\alpha_j<\beta_j\}$. Then

$$S(f,h)=\big(\prod_{j\in J}\mathrm{LM}(g_j)^{\beta_j-\alpha_j}\big)\,g_1^{\alpha_1}\cdots g_s^{\alpha_s}-\big(\prod_{i\in I}\mathrm{LM}(g_i)^{\alpha_i-\beta_i}\big)\,g_1^{\beta_1}\cdots g_s^{\beta_s}.$$

Since $(\prod_{j\in J}g_j^{\beta_j-\alpha_j})\,g_1^{\alpha_1}\cdots g_s^{\alpha_s}=(\prod_{i\in I}g_i^{\alpha_i-\beta_i})\,g_1^{\beta_1}\cdots g_s^{\beta_s}$, we have

$$\begin{aligned}S(f,h)=&\big(\prod_{i\in I}g_i^{\alpha_i-\beta_i}-\prod_{i\in I}\mathrm{LM}(g_i)^{\alpha_i-\beta_i}\big)g_1^{\beta_1}\cdots g_s^{\beta_s}\\ &-\big(\prod_{j\in J}g_j^{\beta_j-\alpha_j}-\prod_{j\in J}\mathrm{LM}(g_j)^{\beta_j-\alpha_j}\big)g_1^{\alpha_1}\cdots g_s^{\alpha_s}.\end{aligned}\tag{6.4}$$

We show that this is a standard expression of $S(f,h)$ by G^t. If at least one of the two summands in the right hand side of (6.4) is 0, then we are done. Otherwise we establish that the two summands have different multidegree. Seeking a contradiction, we suppose that they have the same multidegree. Then since $\prod_{i\in I}\mathrm{LM}(g_i)^{\alpha_i}$ divides the leading monomial of the second summand in the right hand side of (6.4), it also divides the leading monomial of the first summand. Using that for $i\in I$ and $j\notin I$, $\mathrm{LM}(g_i)^{\alpha_i}$ is coprime to $\mathrm{LM}(g_j)^{\beta_j}$, we obtain that

$$\prod_{i\in I}\mathrm{LM}(g_i)^{\alpha_i}\text{ divides }\mathrm{LM}\Big(\big(\prod_{i\in I}g_i^{\alpha_i-\beta_i}-\prod_{i\in I}\mathrm{LM}(g_i)^{\alpha_i-\beta_i}\big)\prod_{i\in I}\mathrm{LM}(g_i)^{\beta_i}\Big).\tag{6.5}$$

We have $\mathrm{LM}\big(\prod_{i\in I}g_i^{\alpha_i-\beta_i}-\prod_{i\in I}\mathrm{LM}(g_i)^{\alpha_i-\beta_i}\big)<_a\prod_{i\in I}\mathrm{LM}(g_i)^{\alpha_i-\beta_i}$, and therefore the degree of the right hand side is less (w.r.t $<_a$) than the degree of the left hand side of (6.5). Since the right hand side is not 0, this is a contradiction. □

Lemma 6.3 and Theorem 6.4 allow us to determine a Gröbner basis for the ideal $\mathbb{I}_t(S)$ associated with a grid S. After these preparations, we can now prove the main results established in this section.

Proof of Theorem 2.2. For each $i\in\underline{n}$, let $g_i:=\prod_{a\in S_i}(x_i-a)$. Let $G=\{g_1,\dots,g_n\}$, and I be the ideal of $\mathbb{K}[x_1,\dots,x_n]$ given by $I:=\langle G\rangle^t$. By Lemma 6.3 all elements of S are t-fold zeros f if and only if $f\in I$. Now we seek to apply Corollary 4.5. For each $i\in\underline{n}$, S_i is λ_i-null and thus the polynomial g_i is λ-lacunary. Therefore, by Lemma 3.2 each polynomial in G^t is λ-lacunary. We will now show that $\boldsymbol{x}^\alpha$ is a (G^t,λ)-stable monomial in f. First, suppose that there is $h\in G^t$ such that $\mathrm{LM}(h)$ divides $\boldsymbol{x}^\alpha$. Let $\beta_1,\dots,\beta_n\in\mathbb{N}_0$ such that $\sum_{i=1}^n\beta_i=t$ and $h=g_1^{\beta_1}\cdots g_n^{\beta_n}$. Then $\mathrm{LM}(h)=\prod_{i=1}^n\mathrm{LM}(g_i)^{\beta_i}=\prod_{i=1}^n x_i^{\beta_i|S_i|}$ and thus, we have $\beta_i|S_i|\le\alpha_i$ for all $i\in\mathbb{N}$, contradicting the assumption stated in (2.3). Therefore the monomial $\mathrm{LM}(h)$ does not divide $\boldsymbol{x}^\alpha$, which establishes Condition (2) of Definition 4.3. Next, we show that f contains no (G^t, λ, α)-shading monomial. Let $\boldsymbol{x}^\gamma \in \mathrm{Mon}(f)$. If $\alpha = \gamma$, then $\boldsymbol{x}^\gamma$ violates Condition (1) of Definition 4.2 and is therefore not (G^t, λ, α)-shading. If $\gamma \neq \alpha$, the assumption yields an $i \in \underline{n}$ such that $\gamma_i \neq \alpha_i$ and (2.4) holds. If $\gamma_i \in [0, \alpha_i - 1]$, then Condition (1) of Definition 4.2 is violated, and so $\boldsymbol{x}^\gamma$ is not (G^t, λ, α)-shading. If $\gamma_i \in [\alpha_i + 1, \alpha_i + \lambda_i]$, then Condition (3) of of Definition 4.2 is violated and thus case $\boldsymbol{x}^\gamma$ is not (G^t, λ, α)-shading. Now we turn to the case that the last alternative in (2.4) holds and that $\gamma_i > \alpha_i + \lambda_i$. Seeking a contradiction, we suppose that Condition (2) of Definition 4.2 is satisfied. This condition tells that there is $\beta \in \mathbb{N}_0^n$ such that $\sum_{j=1}^n \beta_j = t$ and $\mathrm{LM}(g_1^{\beta_1} \dots g_n^{\beta_n}) \mid \boldsymbol{x}^\gamma$ and $\beta_i |S_i| > 0$. Since $\mathrm{LM}(g_1^{\beta_1} \dots g_n^{\beta_n}) \mid \boldsymbol{x}^\gamma$, we have $\beta_j |S_j| \leq \gamma_j$ for all $j \in \underline{n}$. This contradicts the case assumption that the last alternative in (2.4) holds. Hence Condition (2) of Definition 4.2 fails, and thus $\boldsymbol{x}^\gamma$ is not (G^t, λ, α)-shading.

Therefore, f contains no (G^t, λ, α)-shading monomial and thus, $\boldsymbol{x}^\alpha$ is (G^t, λ)-stable. Let $f = \sum_{j=1}^t c_j \, \boldsymbol{x}^{\delta_j} g_{i_j} + r$ be a natural standard expression of f by G. Since $\boldsymbol{x}^\alpha$ is a (G^t, M)-stable monomial in f, Corollary 4.5 yields that $\boldsymbol{x}^\alpha \in \mathrm{Mon}(r)$. Since the leading monomials of the polynomials in G are coprime, Theorem 6.4 yields that the set G^t is a Gröbner basis of I. Since then all elements of I have zero remainder in every standard expression by G^t, we obtain $f \notin I$, and therefore, not all points in $S_1 \times \cdots \times S_n$ are t-fold zeros of f. □

Proof of Corollary 2.3. We assume that $\sum_{i=1}^n \alpha_i \geq \deg(f) - \lambda$, and we show that for $\lambda_1 = \cdots = \lambda_n = \lambda$, the assumptions of Theorem 2.2 are satisfied. Suppose that the assumption on the monomials $\boldsymbol{x}^\gamma \in \mathrm{Mon}(f) \setminus \{\boldsymbol{x}^\alpha\}$ fails. Then there is a monomial $\boldsymbol{x}^\gamma \in \mathrm{Mon}(f)$ with $\gamma \neq \alpha$ such that for all $i \in \underline{n}$, we have $\gamma_i = \alpha_i$ or $\gamma_i > \alpha_i + \lambda$. Then $\deg(f) \geq \deg(\boldsymbol{x}^\gamma) > (\sum_{i=1}^n \alpha_i) + \lambda$, contradicting the assumption (2.5) of Corollary 2.3. Now Theorem 2.2 yields the result. □

7. A Nullstellensatz for grids of multisets

In this section, we add two improvements to Kós and Rónyai's Nullstellensatz for multisets given in [KR12, Theorem 6]: we consider *structured* grids, and we give a generalization in terms of *conditions on the monomials* in the spirit of [Sch08]. Let $f \in \mathbb{K}[x_1, \dots, x_n]$, and let $(m_1, \dots, m_n) \in \mathbb{N}_0^n$. We say that $\boldsymbol{c}$ is a zero with multiplicity vector $(m_1, \dots, m_n)$ if $f(\boldsymbol{x} + \boldsymbol{c}) \in \langle x_1^{m_1}, \dots, x_n^{m_n} \rangle$. Let T be a mapping from the finite set U to $\mathbb{N}_0$. Then we also call T a *multiset* and U the *domain* of T. Let $S_1, \dots, S_n$ be multisets with domains $U_1, \dots, U_n$. For $n \in \mathbb{N}$, the *multigrid* S denoted by $\bigtimes_{i=1} S_i$ is a mapping with domain $U_1 \times \cdots \times U_n$, codomain $\mathbb{N}_0^n$ and $S(u_1, \dots, u_n) := (S_1(u_1), \dots, S_n(u_n))$. Suppose that $U_1, \dots, U_n$ are subsets of a field $\mathbb{K}$. Then we say that $f \in \mathbb{K}[x_1, \dots, x_n]$ *vanishes on the multigrid* S if for all $\boldsymbol{u} = (u_1, \dots, u_n) \in \bigtimes_{i=1}^n U_i$, $\boldsymbol{u}$ is a zero with multiplicity vector $S(\boldsymbol{u}) = (S_1(u_1), \dots, S_n(u_n))$. The following result is a consequence of [KR12, Theorem 1].

Theorem 7.1. *Let S be a multigrid with domain $\bigtimes_{i=1}^n U_i$, and let $f \in \mathbb{K}[x_1, \dots, x_n]$. Then f vanishes on S if and only if f lies in the ideal $\langle g_1, \dots, g_n \rangle$, where $g_i := \prod_{u \in U_i} (x_i - u)^{S_i(u)}$.*

We say that a multiset $S_i : U_i \to \mathbb{N}_0$ is λ_i-null if $\prod_{u \in U_i} (x_i - u)^{S_i(u)}$ is λ_i-lacunary. Now our generalization of [KR12, Theorem 6] is:

Theorem 7.2 (Structured Nullstellensatz for multisets using conditions on the monomials)**.** *Let* $S = \bigtimes_{i=1}^n S_i$ *be a multigrid with domain* $\bigtimes_{i=1}^n U_i$*, let* $f \in \mathbb{K}[x_1, \dots, x_n]$*, and let* $\alpha = (\alpha_1, \dots, \alpha_n)$ *and* $\lambda = (\lambda_1, \dots, \lambda_n)$ *be elements of* $\mathbb{N}_0^n$*. We assume that each* S_i *is* λ_i*-null. Let* $||S_i|| := \sum_{u \in U_i} S_i(u)$*. We assume that* $\alpha_i < ||S_i||$ *for all* $i \in \underline{n}$ *and that* f *contains the monomial* $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$*. Furthermore, we assume that for every monomial* $\boldsymbol{x}^\gamma$ *in* $\mathrm{Mon}(f) \setminus \{\boldsymbol{x}^\alpha\}$*, there is* $i \in \underline{n}$ *such that*

$$\gamma_i \in [0, \alpha_i - 1] \cup [\alpha_i + 1, ||S_i|| - 1] \cup [||S_i||, \alpha_i + \lambda_i]. \tag{7.1}$$

Then f *does not vanish on the multigrid* S*.*

Proof. For each $i \in \underline{n}$, let $g_i := \prod_{u \in U_i} (x_i - u)^{S_i(u)}$. Then the leading monomial of g_i is $x_i^{||S_i||}$. Let I be the ideal of $\mathbb{K}[x_1, \dots, x_n]$ generated by $G = \{g_1, \dots, g_n\}$. By Theorem 7.1, a polynomial f vanishes on the multigrid $S_1 \times \cdots \times S_n$ if and only if it lies in I. The remainder of the proof is an almost verbatim copy of the proof of Theorem 2.1; again, we seek to apply Corollary 4.5. For each $i \in \underline{n}$, S_i is λ_i-null and therefore the polynomial g_i is $(\lambda_1, \dots, \lambda_n)$-lacunary. We will now show that $\boldsymbol{x}^\alpha$ is a (G, λ)-stable monomial in f with respect to $\leq_a$. First, we observe that for each $i \in \underline{n}$ we have $\alpha_i < ||S_i||$ and therefore the monomial $\mathrm{LM}(g_i)$ does not divide $\boldsymbol{x}^\alpha$, which establishes Condition (2) of Definition 4.3. Next, we show that f contains no (G, λ, α)-shading monomial. Let $\boldsymbol{x}^\gamma \in \mathrm{Mon}(f)$. If $\alpha = \gamma$, then $\boldsymbol{x}^\gamma$ violates Condition (1) of Definition 4.2 and is therefore not (G, λ, α)-shading. If $\gamma \neq \alpha$, the assumption yields an $i \in \underline{n}$ such that

$$\gamma_i \in [0, \alpha_i - 1] \cup [\alpha_i + 1, ||S_i|| - 1] \cup [||S_i||, \alpha_i + \lambda_i].$$

If $\gamma_i \in [0, \alpha_i - 1]$, then Condition (1) of Definition 4.2 is violated, and so $\boldsymbol{x}^\gamma$ is not (G, λ, α)-shading. If $\gamma_i \in [\alpha_i + 1, ||S_i|| - 1]$ and Condition (2) of Definition 4.2 is satisfied, then $\mathrm{LM}(g_i) \mid \boldsymbol{x}^\gamma$, and therefore $|S_i| \leq \gamma_i$, contradicting $\gamma_i \leq ||S_i|| - 1$. We conclude that also in the case $\gamma_i \in [\alpha_i + 1, ||S_i|| - 1]$, $\boldsymbol{x}^\gamma$ is not (G, λ, α)-shading. If $\gamma_i \in [||S_i||, \alpha_i + \lambda_i]$, then we have $\gamma_i \geq ||S_i|| > \alpha_i$. If Condition (3) of Definition 4.2 is satisfied, we have $\gamma_i > \alpha_i + \lambda_i$, contradicting $\gamma_i \leq \alpha_i + \lambda_i$. Hence also in this case $\boldsymbol{x}^\gamma$ is not (G, λ, α)-shading. Therefore, f contains no (G, λ, α)-shading monomial. Therefore, $\boldsymbol{x}^\alpha$ is (G, λ)-stable.

Let $f = \sum_{j=1}^t c_j \, \boldsymbol{x}^{\delta_j} g_{i_j} + r$ be a natural standard expression of f by G. Since $\boldsymbol{x}^\alpha$ is a (G, λ)-stable monomial in f, Corollary 4.5 yields that $\boldsymbol{x}^\alpha \in \mathrm{Mon}(r)$. Since the leading monomials of the polynomials in G are coprime, [BW93, Lemma 5.66] yields that the set G is a Gröbner basis of I (cf. [CLO15, p.89, Exercise 11]). Since then all elements of I have zero remainder in every standard expression by G, we obtain $f \notin I$, and therefore, f does not vanish on the multigrid S. □

8. Nullstellensätze for punctured and structured grids

In this Section, we prove Theorem 2.4 and Corollary 2.5. We start with the description of the vanishing ideal of a punctured grid. We note that a similar result that includes multiplicities has been given in [BS09, Theorem 4.1], but our proof is different. For a subset J of $\mathbb{K}[x_1, \dots, x_n]$ and $f \in \mathbb{K}[x_1, \dots, x_n]$, we write $\mathbb{V}(J)$ for the set $\{\boldsymbol{a} \in \mathbb{K}^n \mid f(\boldsymbol{a}) = 0 \text{ for all } f \in J\}$ of common zeros of J.

Theorem 8.1 (cf. [BS09, Theorem 4.1])**.** *Let $X \setminus Y = (\bigtimes_{i=1}^n X_i) \setminus (\bigtimes_{i=1}^n Y_i)$ be a punctured grid with $Y_i \subseteq X_i$ for all $i \in \mathbb{N}$, let $g_i := \prod_{a \in X_i}(x_i - a)$, let $l_i := \prod_{a \in Y_i}(x_i - a)$, and let $f \in \mathbb{K}[x_1, \dots, x_n]$. Then f vanishes on $X \setminus Y$ if and only if $f \in \langle g_1, \dots, g_n, \prod_{i=1}^n \frac{g_i}{l_i} \rangle$. Furthermore, for every admissible monomial ordering $\leq_a$, $\{g_1, \dots, g_n, \prod_{i=1}^n \frac{g_i}{l_i}\}$ is a Gröbner basis with respect to $\leq_a$.*

Proof. We first compute generators of the vanishing ideal $\mathbb{I}(X \setminus Y)$. We have $X \setminus Y = X \setminus \bigcap_{i \in \underline{n}} X_1 \times \cdots \times X_{i-1} \times Y_i \times X_{i+1} \times \cdots \times X_n = X \cap \bigcup_{i \in \underline{n}} X_1 \times \cdots \times X_{i-1} \times (X_i \setminus Y_i) \times X_{i+1} \times \cdots \times X_n = X \cap \bigcup_{i \in \underline{n}} \mathbb{K} \times \cdots \times \mathbb{K} \times (X_i \setminus Y_i) \times \mathbb{K} \times \cdots \times \mathbb{K} = X \cap \bigcup_{i \in \underline{n}} \mathbb{V}(\frac{g_i}{l_i}) = X \cap \mathbb{V}(\prod_{i \in \underline{n}} \frac{g_i}{l_i})$. Now for finite X, Clark's Finitesatz [Cla14, Theorem 7] tells that for any ideal J of $\mathbb{K}[x_1, \dots, x_n]$, we have $\mathbb{I}(X \cap \mathbb{V}(J)) = \mathbb{I}(X) + J$, and therefore

$$\mathbb{I}(X \setminus Y) = \mathbb{I}(X \cap \mathbb{V}(\prod_{i \in \underline{n}} \frac{g_i}{l_i})) = \mathbb{I}(X) + \langle \prod_{i \in \underline{n}} \frac{g_i}{l_i} \rangle = \langle g_1, \dots, g_n, \prod_{i \in \underline{n}} \frac{g_i}{l_i} \rangle.$$

In order to show that $G := \{g_1, \dots, g_n, \prod_{i \in \underline{n}} \frac{g_i}{l_i}\}$ is a Gröbner basis, we again use Buchberger's Characterization Theorem for Gröbner bases [BW93, Theorem 5.64], and we therefore look at the S-polynomials of G. First, we note that for $i, j \in \underline{n}$ with $i \neq j$, the leading monomials of g_i and g_j are coprime, and therefore the S-polynomial of g_i and g_j has a standard expression by $\{g_i, g_j\}$ with remainder 0 ([BW93, Lemma 5.66]); such an expression can also be obtained by setting $t := 1$ in the proof of Theorem 6.4. For checking the other S-polynomials, let $j \in \underline{n}$, and let $p := \prod_{i \in \underline{n}} \frac{g_i}{l_i}$. We compute $S(g_j, p)$. Let $a_i := |X_i|$ and $b_i := |Y_i|$. Then

$$\text{LM}(g_j) = x_j^{a_j} \text{ and } \text{LM}(p) = \prod_{i \in \underline{n}} x_i^{a_i - b_i}.$$

Now let $q := \prod_{i \in \underline{n} \setminus \{j\}} \frac{g_i}{l_i}$; then $q \frac{g_j}{l_j} = p$. Then we have

$$S(g_j, p) = (\prod_{i \in \underline{n} \setminus \{j\}} x_i^{a_i - b_i}) g_j - x_j^{b_j} p = \text{LM}(q)\, g_j - \text{LM}(l_j)\, p.$$

Since $l_j p = q g_j$, we have

$$\text{LM}(q)\, g_j - \text{LM}(l_j)\, p = (l_j - \text{LM}(l_j))\, p - (q - \text{LM}(q))\, g_j.$$

We show that $S(g_j, p) = (l_j - \text{LM}(l_j))\, p - (q - \text{LM}(q))\, g_j$ is a standard expression of $S(g_j, p)$ by $\{g_j, p\}$. If at least one of the summands is 0, we are done. Otherwise, we show that the two summands have different multidegree. Suppose that both summands are nonzero and that they have the same multidegree. Then $\text{LM}(g_j)$ divides $\text{LM}((l_j - \text{LM}(l_j))\, p)$. We have $\deg_{x_j}(\text{LM}(g_j)) = a_j$ and $\deg_{x_j}(\text{LM}((l_j - \text{LM}(l_j))\, p)) = \deg_{x_j}(\text{LM}(l_j - \text{LM}(l_j))\, \text{LM}(p)) = \deg_{x_j}(\text{LM}(l_j - \text{LM}(l_j))) + \deg_{x_j}(\text{LM}(p)) \leq \deg_{x_j}(\text{LM}(l_j - \text{LM}(l_j))) + \deg_{x_j}(p) \leq (b_j - 1) + (a_j - b_j) = a_j - 1$. But then $\text{LM}(g_j) \nmid \text{LM}((l_j - \text{LM}(l_j))\, p)$.

Since all S-polynomials have a standard expression by G with remainder 0, Buchberger's Characterization Theorem yields that G is a Gröbner basis. □

Proof of Theorem 2.4. Let $p := \prod_{i=1}^n \frac{g_i}{l_i}$, and let $G' := \{g_1, \dots, g_n, p\}$. We first show that $\boldsymbol{x}^\alpha$ is a (G', λ)-stable monomial in f. Let us first show that there is no $g \in G'$ with $\text{LM}(g) \mid \boldsymbol{x}^\alpha$. If there is $i \in \underline{n}$ with $\text{LM}(g_i) \mid \boldsymbol{x}^\alpha$, then $|X_i| \leq \alpha_i$, contradicting Assumption (1). If $\text{LM}(p) \mid \boldsymbol{x}^\alpha$, then we have $|X_i| - |Y_i| \leq \alpha_i$ for all $i \in \underline{n}$, contradicting Assumption (2). It remains to show that f contains no (G, λ, α)-shading monomial. Let $\boldsymbol{x}^\gamma \in \mathrm{Mon}(f)$. By the assumptions, there is $i \in \underline{n}$ such that one of the Conditions (3a), (3b), (3c) holds. We fist consider the case that $\gamma_i \in [0, \alpha_i - 1]$. Then Condition (1) of Definition 4.2 fails, and thus $\boldsymbol{x}^\gamma$ is not (G', λ, α)-shading. In the case $\gamma_i \in [\alpha_i + 1, \alpha_i + \lambda_i]$, Condition (3) of Definition 4.2 fails, and thus $\boldsymbol{x}^\gamma$ is not (G', λ, α)-shading. In the case that $\gamma_i \in [\alpha_i + 1, |X_i| - 1]$ and $|X_i| = |Y_i|$, we show that Condition (2) of Definition 4.2 fails. Seeking a contradiction, we suppose that Condition (2) holds. Then there is $g \in G'$ with $\deg_{x_i}(g) > 0$ and $\mathrm{LM}(g) \mid \boldsymbol{x}^\gamma$. Since $|X_i| = |Y_i|$, $\deg_{x_i}(p) = 0$ and thus $g = g_i$. Hence $|X_i| \leq \gamma_i$, contradicting the assumptions. Thus Condition (2) of Definition 4.2 fails, and γ is not (G', λ, α)-shading. Now we consider the case that $\gamma_i \in [\alpha_i + 1, |X_i| - 1]$ and that there is $j \in \underline{n}$ with $\gamma_j < |X_j| - |Y_j|$. Supposing again that γ is (G', λ, α)-shading, we obtain that $\mathrm{LM}(g_i) \mid \boldsymbol{x}^\gamma$ or $\mathrm{LM}(p) \mid \boldsymbol{x}^\gamma$. If $\mathrm{LM}(g_i) \mid \boldsymbol{x}^\gamma$, then $|X_i| \leq \gamma_i$, contradicting the assumptions. If $\mathrm{LM}(p) \mid \boldsymbol{x}^\gamma$, then for all $j \in \underline{n}$, we have $|X_j| - |Y_j| \leq \gamma_j$. This is also excluded by the case assumption.

Therefore $\boldsymbol{x}^\alpha$ is a (G', λ)-stable monomial in f. Let $g_{n+1} := p$ and let $f = \sum_{j=1}^t c_j \, \boldsymbol{x}^{\delta_j} g_{i_j} + r$ be a natural standard expression of f by G'. We note that $g_1, \dots, g_n$ are λ-lacunary by assumption. Furthermore, for each $i \in \underline{n}$, g_i and l_i are λ-lacunary and hence by Lemma 3.2, $p = \prod_{i=1}^n \frac{g_i}{l_i}$ is λ-lacunary. Since $\boldsymbol{x}^\alpha$ is a (G', λ)-stable monomial in f and all $g' \in G'$ are λ-lacunary, Corollary 4.5 yields that $\boldsymbol{x}^\alpha \in \mathrm{Mon}(r)$. By Theorem 8.1, the set G' is a Gröbner basis of $\mathbb{I}(X \setminus Y)$. Since then all elements of $\mathbb{I}(X \setminus Y)$ have zero remainder in every standard expression by G', we obtain $f \notin \mathbb{I}(X \setminus Y)$, and therefore, there is $\boldsymbol{s} \in X \setminus Y$ with $f(\boldsymbol{s}) \neq 0$. □

Proof of Corollary 2.5. We show that for $\lambda_1 = \cdots = \lambda_n = \lambda$, the assumptions of Theorem 2.4 are satisfied. Assume that Assumption (3) of Theorem 2.4 fails. Then there is a monomial $\boldsymbol{x}^\gamma \in \mathrm{Mon}(f) \setminus \{\alpha\}$ such that for all $i \in \underline{n}$, $\gamma_i \notin [0, \alpha_i - 1] \cup [\alpha_i + 1, \alpha_i + \lambda_i]$, which means that for all $i \in \underline{n}$, $\gamma_i = \alpha_i$ or $\gamma_i > \alpha_i + \lambda_i$. Then $\deg(f) \geq \deg(\boldsymbol{x}^\gamma) > (\sum_{i=1}^n \alpha_i) + \lambda$, contradicting Assumption (3) of Corollary 2.5. Now Theorem 2.4 yields the result. □

9. A punctured version of the Alon-Füredi Theorem

In [Cla24], Clark gives a proof of the Alon-Füredi Theorem for counting nonzeros of polynomials on grids that is based on the fact that for a finite set $X \subseteq \mathbb{K}^n$, the vector space dimension of $\mathbb{K}[x_1, \dots, x_n]/\mathbb{I}(X)$ is equal to $|X|$, and this dimension can be determined as the number of monomials $\boldsymbol{x}^\alpha$ that are not leading monomials of any polynomial in $\mathbb{I}(X)$. We proceed by adapting some methods from [Cla24] to punctured grids. For a subset G of $\mathbb{K}[x_1, \dots, x_n]$ and an admissible monomial ordering $\leq_a$, we define two subsets of $\{\boldsymbol{x}^\alpha \mid \alpha \in \mathbb{N}_0^n\}$.

$$\begin{aligned} G\uparrow &:= \{\boldsymbol{x}^\alpha \mid \alpha \in \mathbb{N}_0^n \text{ and } \exists g \in G : \mathrm{LM}(g) \text{ divides } \boldsymbol{x}^\alpha\}, \\ \Delta(G) &:= \{\boldsymbol{x}^\alpha \mid \alpha \in \mathbb{N}_0^n\} \setminus (G\uparrow) = \\ & \{\boldsymbol{x}^\alpha \mid \alpha \in \mathbb{N}_0^n \text{ and there is no } g \in G \text{ such that } \mathrm{LM}(g) \text{ divides } \boldsymbol{x}^\alpha\}. \end{aligned}$$

Theorem 9.1 (Clark's formula, [Cla24]). *Let $\mathbb{K}$ be a field, and let X be a finite subset of $\mathbb{K}^n$, and let $f \in \mathbb{K}[x_1, \dots, x_n]$. Then*

$$|X \setminus \mathbb{V}(f)| = |\Delta(\mathbb{I}(X)) \setminus \Delta(\mathbb{I}(X) + \langle f \rangle)|.$$

Proof. For every finite subset Y of $\mathbb{K}^n$, we have

$$|Y| = |\Delta(\mathbb{I}(Y))|. \tag{9.1}$$

To prove this, let $\varepsilon : \mathbb{K}[x_1, \ldots, x_n] \to \mathbb{K}^Y$, $\varepsilon(p) := \hat{p}|_Y$, where $\hat{p}$ is the function from $\mathbb{K}^n$ to $\mathbb{K}$ induced by p, and $\hat{p}|_Y$ denotes its restriction to Y. Then $\ker \varepsilon = \mathbb{I}(Y)$, and ε is surjective because every mapping from Y to $\mathbb{K}$ can be interpolated by a polynomial. Since ε is a homomorphism of $\mathbb{K}$-vector spaces, we get $\dim_{\mathbb{K}}(\mathbb{K}[x_1, \ldots, x_n]/\mathbb{I}(Y)) = \dim_{\mathbb{K}}(\mathbb{K}^Y) = |Y|$. The vector space dimension of the residue class ring $\dim_{\mathbb{K}} \mathbb{K}[x_1, \ldots, x_n]/\mathbb{I}(Y)$ is equal to $|\Delta(\mathbb{I}(Y))|$ because $\{\boldsymbol{x}^\alpha + \mathbb{I}(Y) \mid \boldsymbol{x}^\alpha \in \Delta(\mathbb{I}(Y))\}$ is a basis of this vector space. This proves (9.1). Hence $|X \cap \mathbb{V}(f)| = |\Delta(\mathbb{I}(X \cap \mathbb{V}(f)))|$. By Clark's Finitesatz [Cla14, Theorem 7], the last expression is equal to $|\Delta(\mathbb{I}(X) + \langle f \rangle)|$. Thus $|X \setminus \mathbb{V}(f)| = |X| - |X \cap \mathbb{V}(f)| = |\Delta(\mathbb{I}(X))| - |\Delta(\mathbb{I}(X) + \langle f \rangle)|$. Since $\mathbb{I}(X) \subseteq \mathbb{I}(X) + \langle f \rangle$, we have $\Delta(\mathbb{I}(X) + \langle f \rangle) \subseteq \Delta(\mathbb{I}(X))$, and therefore $|\Delta(\mathbb{I}(X))| - |\Delta(\mathbb{I}(X) + \langle f \rangle)| = |\Delta(\mathbb{I}(X)) \setminus \Delta(\mathbb{I}(X) + \langle f \rangle)|$. ☐

For $a, b \in \mathbb{N}_0$, we will denote the interval $\{x \in \mathbb{N}_0 \mid a \le x < b\}$ by $[a, b)$.

Lemma 9.2. *Let $X = \bigtimes_{i=1}^n X_i, Y = \bigtimes_{i=1}^n Y_i$ be grids over the field $\mathbb{K}$ with $Y_i \subseteq X_i$ for all $i \in \underline{n}$, and for each $i \in \underline{n}$, let $a_i := |X_i|$ and $b_i := |Y_i|$. Then $\Delta(\mathbb{I}(X \setminus Y)) = \{\boldsymbol{x}^\alpha \mid \alpha \in \bigtimes_{i \in \underline{n}}[0, a_i) \setminus \bigtimes_{i \in \underline{n}}[a_i - b_i, a_i)\}$.*

Proof. Let $g_i := \prod_{a \in X_i}(x_i - a)$, and $l_i := \prod_{a \in Y_i}(x_i - a)$. Then by Theorem 8.1, $G' := \{g_1, \ldots, g_n, \prod_{i=1}^n \frac{g_i}{l_i}\}$ is a Gröbner basis for $\mathbb{I}(X \setminus Y)$. Hence $\Delta(\mathbb{I}(X \setminus Y)) = \Delta(G') = \Delta(\{g_1, \ldots, g_n\}) \cap \Delta(\{\prod_{i=1}^n \frac{g_i}{l_i}\}) = \Delta(\{g_1, \ldots, g_n\}) \setminus \{\prod_{i=1}^n \frac{g_i}{l_i}\}{\uparrow} = \{\boldsymbol{x}^\alpha \mid \alpha \in \bigtimes_{i \in \underline{n}}[0, a_i) \setminus \bigtimes_{i \in \underline{n}}[a_i - b_i, a_i)\}$. ☐

Theorem 9.3 (Clark's Monomial Alon-Füredi Theorem [Cla24])**.** *Let X be a finite subset of $\mathbb{K}^n$, let $f \in \mathbb{K}[x_1, \ldots, x_n]$, and let $g \in \mathbb{I}(X) + \langle f \rangle$ with $g \neq 0$. Then*

$$|X \setminus \mathbb{V}(f)| \ge |\Delta(\mathbb{I}(X)) \cap \{\textsc{Lm}(g)\}{\uparrow}| .$$

Proof. Since $g \in \mathbb{I}(X) + \langle f \rangle$, g vanishes on $\mathbb{V}(f) \cap X$, and thus $X \cap \mathbb{V}(f) \subseteq X \cap \mathbb{V}(g)$, and therefore $X \setminus \mathbb{V}(f) \supseteq X \setminus \mathbb{V}(g)$. Now by Theorem 9.1, we have $|X \setminus \mathbb{V}(g)| = |\Delta(\mathbb{I}(X)) \setminus \Delta(\mathbb{I}(X) + \langle g \rangle)|$. In addition, $\Delta(\mathbb{I}(X)) \setminus \Delta(\mathbb{I}(X) + \langle g \rangle) = \Delta(\mathbb{I}(X)) \cap (\mathbb{I}(X) + \langle g \rangle){\uparrow} \supseteq \Delta(\mathbb{I}(X)) \cap \{g\}{\uparrow} = \Delta(\mathbb{I}(X)) \cap \{\textsc{Lm}(g)\}{\uparrow}$. Altogether, $|X \setminus \mathbb{V}(f)| \ge |\Delta(\mathbb{I}(X)) \cap \{\textsc{Lm}(g)\}{\uparrow}|$. ☐

Theorem 9.4 (Alon-Füredi-Clark for punctured grids)**.** *Let $X = \bigtimes_{i=1}^n X_i, Y = \bigtimes_{i=1}^n Y_i$ be grids over the field $\mathbb{K}$ with $Y_i \subseteq X_i$ for all $i \in \underline{n}$, let $P := X \setminus Y$, let let $f \in \mathbb{K}[x_1, \ldots, x_n]$, and for $i \in \underline{n}$, let $a_i := |X_i|$ and $b_i := |Y_i|$, and let $\le_a$ be an admissible monomial ordering. Let $g \in \mathbb{I}(P) + \langle f \rangle$ with $\textsc{Lm}(g) = x_1^{e_1} \cdots x_n^{e_n}$ and $e_i < a_i$ for all $i \in \underline{n}$. Then*

$$|P \setminus \mathbb{V}(f)| \ge \prod_{i=1}^n (a_i - e_i) - \prod_{i=1}^n \min(b_i, a_i - e_i).$$

Proof. By Theorem 9.3, we have

$$|P \setminus \mathbb{V}(f)| \ge |\Delta(\mathbb{I}(P)) \cap \{\textsc{Lm}(g)\}{\uparrow}|.$$

By Lemma 9.2, we have $\Delta(\mathbb{I}(P)) \cap \{\mathrm{LM}(g)\}\uparrow = \{x^\alpha \mid \alpha \in E\}$, where E is given by

$$E = (\bigtimes_{i\in\underline{n}}[0, a_i) \setminus \bigtimes_{i\in\underline{n}}[a_i - b_i, a_i)) \cap \bigtimes_{i\in\underline{n}}[e_i, a_i).$$

Then we have

$$E = \bigtimes_{i\in\underline{n}}[e_i, a_i) \setminus \bigtimes_{i\in\underline{n}}[a_i - b_i, a_i)) = \bigtimes_{i\in\underline{n}}[e_i, a_i) \setminus (\bigtimes_{i\in\underline{n}}[e_i, a_i) \cap \bigtimes_{i\in\underline{n}}[a_i - b_i, a_i))$$
$$= \bigtimes_{i\in\underline{n}}[e_i, a_i) \setminus \bigtimes_{i\in\underline{n}}[\max(e_i, a_i - b_i), a_i).$$

Since $a_i - \max(e_i, a_i - b_i) = \min(a - e_i, b_i)$, we can now compute $|E| = \prod_{i=1}^n (a_i - e_i) - \prod_{i=1}^n \min(b_i, a_i - e_i)$. □

We note that similar to Clark's version for unpunctured grids in [Cla24], this lower bound can be attained for every choice of $(e_1, \dots, e_n)$ with $e_i < a_i$ for all $i \in \underline{n}$.

Proposition 9.5. *Let $X = \bigtimes_{i=1}^n X_i, Y = \bigtimes_{i=1}^n Y_i$ be grids over the field $\mathbb{K}$ with $Y_i \subseteq X_i$ for all $i \in \underline{n}$, and let $P := X \setminus Y$. For each $i \in \underline{n}$, let $a_i := |X_i|$, $b_i := |Y_i|$, let $e_i \in [0, a_i - 1)$ and let $E_i \subseteq X_i$ be such that $|E_i| = e_i$ and ($E_i \subseteq X_i \setminus Y_i$ or $X_i \setminus Y_i \subseteq E_i$). Let $f_i := \prod_{a\in E_i}(x_i - a)$ and $f := \prod_{i=1}^n f_i$. Then for every admissible monomial ordering, $\mathrm{LM}(f) = x_1^{e_1} \cdots x_n^{e_n}$, and $|P \setminus \mathbb{V}(f)| = \prod_{i=1}^n (a_i - e_i) - \prod_{i=1}^n \min(b_i, a_i - e_i)$.*

Proof. It is easy to see that $\mathrm{LM}(f) = x_1^{e_1} \cdots x_n^{e_n}$ and that the nonzeros of f on X are given by $X \setminus \mathbb{V}(f) = \bigtimes_{i=1}^n (X_i \setminus E_i)$, and hence $|X \setminus \mathbb{V}(f)| = \prod_{i=1}^n (a_i - e_i)$. We will now compute how many of these nonzeros lie in Y. To this end, we observe that

$$\bigtimes_{i=1}^n (X_i \setminus E_i) \cap \bigtimes_{i=1}^n Y_i = \bigtimes_{i=1}^n (Y_i \setminus E_i).$$

In the case that $E_i \subseteq X_i \setminus Y_i$, we have $Y_i \setminus E_i = Y_i$, and therefore $|Y_i \setminus E_i| = b_i$. In the case that $X_i \setminus Y_i \subseteq E_i$, we have $X_i \setminus E_i = (Y_i \cup (X_i \setminus Y_i)) \setminus E_i = (Y_i \setminus E_i) \cup ((X_i \setminus Y_i) \setminus E_i) = (Y_i \setminus E_i) \cup \varnothing = Y_i \setminus E_i$, and therefore $|Y_i \setminus E_i| = a_i - e_i$. In each of these cases, $|Y_i \setminus E_i| = \min(a_i - e_i, b_i)$. Therefore, $|P \setminus \mathbb{V}(f)| = |X \setminus \mathbb{V}(f)| - |Y \setminus \mathbb{V}(f)| = \prod_{i=1}^n (a_i - e_i) - \prod_{i=1}^n \min(b_i, a_i - e_i)$. □

Based on Theorem 9.4, we can now prove Theorem 2.6.

Proof of Theorem 2.6. We fix an admissible ordering $\leq_a$ such that $\sum_{i=1}^n \gamma_i < \sum_{i=1}^n \delta_i$ implies $\boldsymbol{x}^\gamma \leq_a \boldsymbol{x}^\delta$.

For proving (1), we divide f by G', where G' is the Gröbner basis of $\mathbb{I}(P)$ given in Theorem 8.1 and obtain a standard expression $f = \sum_{i=1}^{n+1} h_i g_i' + r$ such that the remainder r contains no monomial divisible by a $\mathrm{LM}(g')$ with $g' \in G'$. If $r = 0$, then $f \in \mathbb{I}(P)$, and thus $\mathbb{I}(P) \setminus \mathbb{V}(f) = \varnothing$. If $r \neq 0$, then $\deg(r) \leq \deg(f)$. Let $(e_1, \dots, e_n) := \mathrm{LEXP}(r)$. Now Theorem 9.4 for $g := r$ yields $|P \setminus \mathbb{V}(f)| \geq \prod_{i=1}^n (a_i - e_i) - \prod_{i=1}^n \min(b_i, a_i - e_i)$. Set $y_i := a_i - e_i$. Then $1 \leq y_i \leq a_i$. If $y_i \leq b_i$ for all $i \in \underline{n}$, then $a_i - b_i \leq e_i$ for all $i \in \underline{n}$, and thus $\mathrm{LM}(r)$ is divisible by the leading monomial $\prod_{i\in\underline{n}} x_i^{a_i - b_i}$ of one of the elements of G'. Hence there is $i \in \underline{n}$ with $y_i > b_i$. Now $\sum_{i=1}^n y_i = \sum_{i=1}^n (a_i - e_i) = \sum_{i=1}^n a_i - \sum_{i=1}^n e_i = \sum_{i=1}^n a_i - \deg(r) \geq \sum_{i=1}^n a_i - \deg(f)$. Hence $(y_1, \dots, y_n) \in A$, which proves (2.6).

For proving (2), let $(e_1, \ldots, e_n) := \textsc{Lexp}(f)$. Now Theorem 9.4 for $g := f$ yields $|P \setminus \mathbb{V}(f)| \geq \prod_{i=1}^{n}(a_i - e_i) - \prod_{i=1}^{n} \min(b_i, a_i - e_i)$. Set $y_i := a_i - e_i$. Then since $e_i \leq \deg_{x_i}(f)$, we have $a_i - \deg_{x_i}(f) \leq y_i \leq a_i$. Now $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n}(a_i - e_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} e_i = \sum_{i=1}^{n} a_i - \deg(f)$. Hence $(y_1, \ldots, y_n) \in B$, which proves (2.7). □

Setting $Y_i := \varnothing$ for all $i \in \underline{n}$, one recovers the classical Alon-Füredi Theorem for grids.

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