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A note on minimum $K_{2,3}$-saturated graphs

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Abstract

A graph G is said to be $K_{2,3}$-saturated if G contains no copy of $K_{2,3}$ as a subgraph, but for any edge e in the complement of G the graph $G+e$ does contain a copy of $K_{2,3}$. The minimum number of edges of a $K_{2,2}$-saturated graph of given order n was precisely determined by Ollmann in 1972. Here, we determine the asymptotic behavior for the minimum number of edges in a $K_{2,3}$-saturated graph.

1 Introduction

We denote the complete graph on t vertices by K_t, and the complete bipartite graph with partite sets of size a and b by $K_{a,b}$. We let $G = (V, E)$ be a graph on $|V| = n$ vertices and $|E|$ edges. The graph G is said to be *F-saturated* if G contains no copy of F as a subgraph, but for any edge e in the complement of G, the graph $G + e$ contains a copy of F, where $G + e$ denotes the graph $(V, E \cup \{e\})$. For a graph F we will denote the minimum size of an F-saturated graph by $\mathrm{sat}(n, F)$. In 1964 Erdős, Hajnal and Moon [3] determined $\mathrm{sat}(n, K_t)$ for all n, t. Determining the exact value of this function for a given graph F is quite difficult in general, and the

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sat-function is known for relatively few graphs. The value of $\mathrm{sat}(n, K_{2,2})$ was shown to be $\lfloor \frac{3n-5}{2} \rfloor$ by Ollmann [6], and a shorter proof was later given by Tuza [7]. (See also Fisher, Fraughnaugh, and Langley [4] for a strengthening of Ollmann's result.) Bryant and Fu [2] studied minimum $K_{2,2}$-saturated graphs in the class of bipartite graphs. Kászonyi and Tuza in [5] give a general upper bound for $\mathrm{sat}(n, F)$ which is sharp in many cases. For a survey of related results see [1].

In this note we determine the asymptotic behavior of $\mathrm{sat}(n, K_{2,3})$.

Theorem 1 *There is a constant C such that for all $n \geq 5$ we have*

$$2n - Cn^{3/4} \leq \mathrm{sat}(n, K_{2,3}) \leq 2n - 3. \tag{1}$$

2 Proof of Theorem 1

The following construction, which can be obtained from the argument in [5], shows the upper bound in (1). Let G' be either the disjoint union of a 2-regular $K_{2,2}$-free graph on $n-2$ vertices and a single vertex, or the disjoint union of a 2-regular $K_{2,2}$-free graph on $n-3$ vertices and a single edge. Let G be the *join* of a single vertex v and the graph G', that is, we add to G' the vertex v and all edges (v, u) with $u \in V(G')$. As G' is $K_{2,2}$- and $K_{1,3}$-free the graph G is $K_{2,3}$-free. On the other hand, any edge added to G creates a $K_{1,3}$ in G' and thus creates a $K_{2,3}$ in G. This proves the upper bound in (1).

We now show the lower bound. Some of the ideas come from [4]. Let G be a minimum $K_{2,3}$-saturated graph on $[n] = \{1, \ldots, n\}$.

If $\delta(G) \geq 4$, then $e(G) \geq 2n$ and we are done. Thus, assume that $\delta(G) \leq 3$.

Let $a \in [n]$ be a vertex of minimum degree. Note that G has diameter at most three. Take a breadth-first search tree T starting at a. Let its levels be $V_1 \cup V_2 \cup V_3 = [n] \setminus \{a\}$, where the distance from every $x \in V_i$ to a is i. Let R be the graph with the edge set $E(G) \setminus E(T)$. Let $v_i = |V_i|$ and $|e(V_i)| = e_i$. We use $G[A]$ to denote the subgraph of G induced by A, and $G[A, B]$ to denote the bipartite subgraph of G containing all edges with one end-vertex in each of A and B.

Partition $V_3 = Y_0 \cup Y_1 \cup Y_2$, where Y_2 consists of all vertices sending at least two edges to V_2, Y_1 consists of all vertices of $V_3 \setminus Y_2$ which are connected to some vertex of Y_2 by a path in $G[V_3]$, and $Y_0 = V_3 \setminus (Y_2 \cup Y_1)$. Clearly, there are no edges between $Y_2 \cup Y_1$ and Y_0. Let $y_i = |Y_i|$.

If $\delta(G) = 1$, say $\Gamma(a) = \{b\}$, then for any $x \in [n] \setminus \{a, b\}$, x and b have at least two common neighbors. (Indeed, consider adding the edge (x, a).) Thus, $e(R[V_2]) \geq v_2$, $Y_2 = V_3$, and $e(R[V_2, V_3]) \geq 2v_3 - e(T[V_2, V_3]) \geq v_3$. We obtain the required bound:

$$e(G) = e(T) + e(R) \geq n - 1 + v_2 + v_3 = 2n - 3.$$

Thus we we can assume that $2 \leq \delta(G) \leq 3$.

Claim 1 $G[Y_0]$ has at most one component which is a tree.

Proof of Claim. Here and further on, we will refer to the vertices in the smaller partite set of $K_{2,3}$ as being red, and those in the larger set as blue.

Suppose that the claim is not true and let L and M be distinct tree components of $G[Y_0]$. Let l_1, m_1 be some leaves of L and M respectively. Furthermore, denote the (unique) l_1's neighbor in V_2 by l_0, and denote the (unique) m_1's neighbor in V_2 by m_0.

As we have assumed that $\delta(G) > 1$, $G[Y_0]$ has no isolated vertices - so denote l_1's neighbor in L by l_2, and do similarly for m_1. Consider adding the edge (l_1, m_1) to G. Without loss of generality assume that l_1 is red and m_1 is blue. Then it must be the case that m_1, l_0, l_2 are blue. The other red vertex must be in V_2 and thus it must be m_0. Thus (l_0, m_0) and (m_0, l_2) are edges of G.

Suppose we have already constructed $l_1, \ldots, l_i$ such that they span a path in L and each of $l_2, \ldots, l_i$ is adjacent to m_0. Consider $G + (m_1, l_i)$. If m_1 is red then it must be that l_i, m_0, and m_2 are blue. The other red vertex must thus be in V_2 and adjacent to l_i. But this would imply that l_i has two neighbors in V_2, a contradiction to the definition of Y_0. Thus m_1 is blue. The vertex m_2 cannot be red as l_i and its red partner must have two common neighbors in G. Thus m_0 is red and l_i must have at least two neighbors in L, both adjacent to m_0. Let l_{i+1} be such a neighbor different from l_{i-1}. We have enlarged the sequence to $l_1, \ldots, l_{i+1}$.

This process must stop at some point (since all vertices $l_1, \ldots, l_i$ are pairwise distinct), which gives us the desired contradiction. ∎

Hence, the number of edges of R which are incident to V_3 is

$$e(R[V_2, V_3]) + e(R[V_3]) \geq y_2 + y_1 + y_0 - 1. \tag{2}$$

Partition $V_2 = X_0 \cup X_1 \cup X_2$, where X_2 consists of those vertices which send at least two edges to V_1, X_1 consists of those vertices from $V_2 \setminus X_2$ which are connected by a path in $G[V_2]$ to a vertex of X_2, and let X_0 consist of the remaining vertices of V_2, that is $X_0 = V_2 \setminus (X_2 \cup X_1)$. Thus, $G[X_2 \cup X_1, X_0]$ is empty. Let $x_i = |X_i|$. Recall that a is a vertex of minimum degree. Let us denote its neighbors by $b_1, \ldots, b_{\deg(a)}$. Let $\mathcal{T}_1$ be the set of trees of $G[X_0]$ each of which contains a leaf vertex (in $G[V_2]$) that shares an edge with b_1. Furthermore, for $2 \leq i \leq \deg(a)$ let $\mathcal{T}_i$ be the set of trees of $G[X_0]$ each of which contains a leaf vertex that shares an edge with b_i and are *not* in $\cup_{j=1}^{i-1} \mathcal{T}_j$. We denote $|\mathcal{T}_i| = t_i$.

It follows that

$$e(R[V_1 \cup V_2]) \geq x_2 + x_1 + x_0 - \Sigma_{i=1}^{\deg(a)} t_i. \tag{3}$$

Claim 2 Let j be fixed and consider any two distinct tree components, T_1, T_2, in $\mathcal{T}_j$. Then any two such trees are connected via a path of length at most three through V_3.

Proof of Claim. Let l_1, l_2 be leaves of T_1, T_2, respectively, such that (l_1, b_j) and (l_2, b_j) are edges of G. Denote l_i's adjacency in T_i by m_i, $i = 1, 2$. We use the fact that l_1 and l_2 are leaf vertices in T_1 and T_2, respectively. Consider the graph $G + (l_1, l_2)$, and without loss of generality, we may assume that in the copy of $K_{2,3}$ formed l_1 is red and l_2 is blue. If no vertex of V_3 is used in the $K_{2,3}$ formed upon the addition of edge (l_1, l_2), then it must be the case that the copy of $K_{2,3}$ sits on the set of vertices $\{b_j, l_1, l_2, m_1, m_2\}$, as these are the only neighbors of either l_1 or l_2 outside of V_3. It would then follow that m_1 and m_2 are both colored blue, and we would reach a contradiction as the edge (l_1, m_2) is not in $E(G)$. Thus, some vertex in V_3, say z, must be in the copy of $K_{2,3}$. If b_j is also in the copy of $K_{2,3}$, then b_j and z must both be blue, and thus m_2 is red. This would force the edges (l_1, z) and (m_2, z) to exist in G and the claim would hold. Otherwise it must be the case that a vertex in V_3 is used and no vertex in V_1 is used. As l_1 and l_2 must lie on a C_4 in $G + (l_1 l_2)$, and no edges other than (l_1, l_2) exist between T_1 and T_2, the claim holds. ∎

We claim that this allows us to add an extra term of $t_j - o(n)$ to the right-hand side of (2) for each $j \in [\deg(a)]$. Let $V_3 = \{u_1, \ldots, u_m\}$, where $m = v_3$. Let j be fixed, $V(\mathcal{T}_j)$ denote the set of vertices contained in the trees of $\mathcal{T}_j$ and let $d_i = d_{V(\mathcal{T}_j)}(u_i)$, $i \in [m]$, the number of G-neighbors of u_i in $V(\mathcal{T}_j)$. If $e(R[V_3]) \geq 2n$, then $e(G) \geq 2n$ and we are done; so assume $e(R[V_3]) < 2n$.

Observe that in (2), we counted at most one edge of $R[V_2, V_3]$ per every vertex in V_3. Hence, the following is true:

$$e(R[V_2, V_3]) + e(R[V_3]) \geq y_2 + y_1 + y_0 - 1 + \sum_{i=1}^{m} (d_i - 2)_+,$$

where $f_+ = f$ if $f \geq 0$ and $f_+ = 0$ otherwise. Assume that $d_1 \geq d_2 \geq \cdots \geq d_m$. Let $k \in [m]$ be the largest index such that $d_k \geq n^{1/4}$. Let $T_1, \ldots, T_{t_j}$ be the trees of $\mathcal{T}_j$. Let A consist of those indices $i \in [t_j]$ such that G has no edges between the tree T_i and $\{u_1, \ldots, u_k\}$. We have $|A| \geq t_j - d_1 - \cdots - d_k$. The definition of A and Claim 2 imply that any two trees T_p, T_q with $p, q \in A$ must be connected in G via $\{u_{k+1}, \ldots, u_m\}$ by a path of length at most three. But each u_i can serve at most $\binom{d_i}{2}$ pairs p, q. Furthermore, each edge of $R[\{u_{k+1}, \ldots, u_m\}]$ serves at most $(n^{1/4})^2$ pairs. Hence,

$$\binom{|A|}{2} \leq mn^{1/2} + 2n \cdot n^{1/2} \leq 3n^{3/2},$$

that is, $d_1 + \cdots + d_k \geq t_j - |A| \geq t_j - O(n^{3/4})$. As $d_i - 2 \geq (1 - 2n^{-1/4}) d_i$ if $d_i > n^{1/4}$, we conclude that $\sum_{i=1}^{m} (d_i - 2)_+ \geq t_j - O(n^{3/4})$, that is,

$$e(R[V_2, V_3]) + e(R[V_3]) \geq y_2 + y_1 + y_0 + t_j - O(n^{3/4}).$$

Moreover, we can do this for all j, $1 \leq j \leq \deg(a)$. Note that the improvement of $t_j - O(n^{3/4})$ comes by considering $G[V(\mathcal{T}_j), V_3]$ and that $V(\mathcal{T}_i) \cap V(\mathcal{T}_j) = \emptyset$ for distinct $i, j \in [\deg(a)]$. Hence, we obtain a further strengthening, that is,

$$e(R[V_2,V_3]) + e(R[V_3]) \geq y_2 + y_1 + y_0 + \Sigma_{i=1}^{\deg(a)} t_i - O(n^{3/4}). \tag{4}$$

By (3) and (4) we have

$$e(G) = e(T) + e(R) \geq n + x_1 + x_2 + x_0 + y_2 + y_1 + y_0 - O(n^{3/4}).$$

As $x_0+x_1+x_2+y_0+y_1+y_2 = n-(\delta(G)+1)$, we conclude that $e(G) \geq 2n-O(n^{3/4})$. ∎

3 Concluding Remarks

Unfortunately, we were not able to obtain an exact result for $K_{2,3}$, nor the asymptotic of the next interesting case, $\mathrm{sat}(n, K_{3,3})$. We conjecture that $\mathrm{sat}(n, K_{3,3}) = (3 + o(1))n$, where the upper bound comes from applying twice the join operation to a $K_{2,2}$-free 2-regular graph on $n-2$ vertices.

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