Repeatedly applying the Combinatorial Nullstellensatz for Zero-sum Grids to Martin Gardner's minimum no-3-in-a-line problem

Seunghwan Oh*, John R. Schmitt†, Xianzhi Wang ‡

Abstract

A 1976 question of Martin Gardner asked for the minimum size of a placement of queens on an $n \times n$ chessboard that is maximal with respect to the property of 'no-3-in-a-line'. The work of Cooper, Pikhurko, Schmitt and Warrington showed that this number is at least n in the cases that $n \not\equiv 3 \pmod 4$, and at least $n-1$ in the case that $n \equiv 3 \pmod 4$. When $n > 1$ is odd, Gardner conjectured the lower bound to be $n+1$. We prove this conjecture in the case that $n \equiv 1 \pmod 4$. The proof relies heavily on a recent advancement to the Combinatorial Nullstellensatz for zero-sum grids due to Bogdan Nica.

Keywords: Combinatorial Nullstellensatz, chessboard, zero-sum grid, Martin Gardner

1 Introduction.

The journal *Scientific American* was the home to Martin Gardner's Mathematical Games column for over 25 years. In one such column in 1976, he gave an intriguing question in extremal combinatorics. Gardner asked for the minimum number of counters one can put on an $n \times n$ chessboard so that upon the addition of one more counter, there are three in a line and there was not three in a line prior to the addition (see Gardner [6], Chapter 5, pg. 71). This is called the *minimum no-3-in-a-line problem.* We might allow for 'line' to be a straight line of any slope. In this case, we point the reader to the recent work of Aichholzer, Eppstein, and Hainzl [1]. However, in this paper we examine when 'line' refers only to orthogonals and diagonals (as

*Department of Mathematics, Middlebury College, Middlebury, VT 05753.

†Department of Mathematics, Middlebury College, Middlebury, VT 05753. Supported in part by Fulbright Austria. Corresponding author: jschmitt@middlebury.edu.

‡Department of Mathematics, Texas A&M University, College Station, TX 77843-3368.

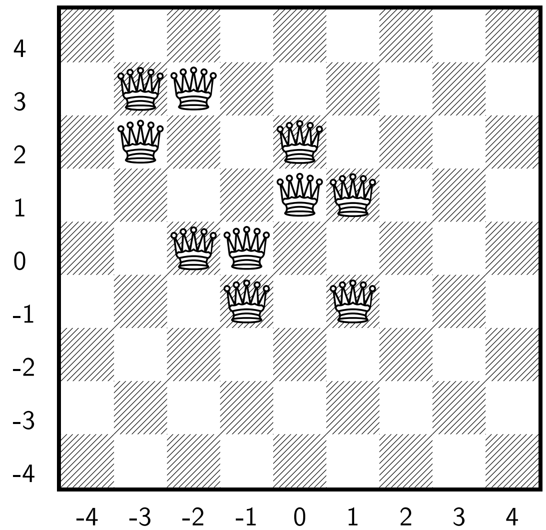

Figure 1: An example of a maximal placement with 10 queens on a 9×9 board.

was also done in [3]); this is the *queens version* of the problem since it properly relates to how a queen is permitted to move in a game of chess.

Consider Figure 1, which shows a 9×9 chessboard with a placement of 10 queens with no three in a line. This is a *maximal* placement since if we add one more queen to any vacant square we will create three in a line. For example, an additional queen placed in the top left corner produces three in a line along the main diagonal. There are no placements of nine or fewer queens on the 9×9 board, making the one given of minimum size (where *size* of a placement is the number of queens in the placement).

A history of the queens version problem is given in [3][1]. Here we offer a brief summary of known results. Cooper, Pikhurko, Schmitt and Warrington [3] gave the following.

Theorem 1. *[Cooper, Pikhurko, Schmitt, Warrington [3]] For $n \geq 1$, the answer to Gardner's no-3-in-a-line queens version problem is at least n, except in the case when n is congruent to* 3 *modulo* 4*, in which case one less may suffice.*

A simple combinatorial proof in the case of n even was given, though this only provided a lower bound of $n-1$ when n is odd. To obtain the full result, the proof of Theorem 1 relied on Alon's Combinatorial Nullstellensatz (reproduced here as Theorem 3).[2]

Let $m_3(n)$ denote the answer to Gardner's no-3-in-a-line queens version problem on an $n \times n$

[1]The published version of this paper contains an editorial error in the abstract. The arxiv version 2 corrects this error.

[2]Alon's seminal work [2] contains many applications of this theorem, and these have served as a huge inspiration to many, including ourselves. This inspiration continues into this present paper.

n	1	2	3	4	5	6	7	8	9
$m_3(n)$	1	4	4	4	6	6	8	9	10
n	10	11	12	13	14	15	16	17	18
$m_3(n)$	10	12	12	14	15	16	17	18	18
n	19	20	21	22	23	24	25	26	27
$m_3(n)$	20	21	22	23	24	25	26	26	28

Table 1: All known values of $m_3(n)$.

chessboard. Gardner knew the precise value of $m_3(n)$ for small values of n, and a few more precise values were given in [3]. Subsequently, Rob Pratt [10] framed the problem as an integer linear programming problem and Don Knuth [8] brought the power of SAT-solvers to bear on the problem. Their results are cumulatively stated as an entry in the On-line Encyclopedia of Integer Sequences [11, A219760]. All values for which $m_3(n)$ is known precisely are given in Table 1.

The data given in Table 1 suggested to us that for n odd and $n \geq 3$ that we should have $m_3(n) \geq n+1$; Gardner posited the same [5][3]. We prove the conjecture in the case that n is congruent to 1 modulo 4 as our main result.

Theorem 2. *For n congruent to 1 modulo 4 and $n \geq 5$, we have $m_3(n) \geq n+1$.*

The proof of our main result relies on a mixture of the polynomial method, combinatorial arguments and linear algebra, some of which are similar to those found in [3]. However, a new, key ingredient is a recent strengthening of Alon's Combinatorial Nullstellensatz due to Bogdan Nica [9]. Before stating Nica's theorem, we recall Part II of the Combinatorial Nullstellensatz (i.e., the Non-vanishing Corollary).

Theorem 3. *[Combinatorial Nullstellensatz, Theorem 1.2 [2]] Let F be an arbitrary field, and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $F[x_1, \ldots, x_n]$. Suppose the degree $\deg(f)$ of f is $\sum_{i=1}^n t_i$, where each t_i is a non-negative integer, and suppose the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in f is nonzero. Then, if $S_1, \ldots, S_n$ are subsets of F with $|S_i| > t_i$, there are $s_1 \in S_1, \ldots, s_n \in S_n$ so that $f(s_1, \ldots, s_n) \neq 0$.*

We say that a subset S of a field F is a *zero-sum* if the sum of the elements in S is the zero of the field. In the case that each of the S_i is a zero-sum subset of the field, Nica [9] showed that

[3] In a typed letter that Martin Gardner wrote to John H. Conway dated 2 June 1975, he wrote, "It would be nice if the minimum for all odd n were $n+1$, and for all even n, n or $n+2$."

the same conclusion in Theorem 3 holds under a slightly weaker degree restriction placed upon the degree of the polynomial. Nica's result is as follows.

Theorem 4. *[Combinatorial Nullstellensatz for Zero-sum Grids, [9]] Let F be an arbitrary field, and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $F[x_1, \ldots, x_n]$. Suppose the degree* $\deg(f)$ *of f is $1 + \sum_{i=1}^{n} t_i$, where each t_i is a non-negative integer, and suppose the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in f is nonzero. Then, if $S_1, \ldots, S_n$ are zero-sum subsets of F with $|S_i| > t_i$, there are $s_1 \in S_1, \ldots, s_n \in S_n$ so that $f(s_1, \ldots, s_n) \neq 0$.*

This Nullstellensatz for Zero-sum Grids is, in fact, a particular case of a more expansive theorem that we won't discuss here. Roughly, it says that we may relax the degree constraints on the polynomial to reach the same conclusion of the Combinatorial Nullstellensatz whenever the grid is "structured", see [9]. As far as we know, the proof of Theorem 2 represents the first application of this particular generalization of the Combinatorial Nullstellensatz [9] outside of that paper. It is interesting to us whenever a generalization of Alon's Nullstellensatz is truly needed for a combinatorial problem, as is the case for our main result.

This paper is organized as follows. In Section 2 we give the necessary definitions and a sketch of the proof of Theorem 2. In Section 3 we prove Theorem 2. In Section 4 we provide some further remarks and also include insights into the difficulty of showing $m_3(n) \geq n + 1$ for the case when n is 3 modulo 4.

2 Definitions and notation

We adopt many of the definitions and notation originally used in [3], with one important exception. Let the infinite square $\mathbb{Z}$-lattice be called a *chessboard* and its vertices be called *squares*. A *board* B is a finite subset of the chessboard. For n odd, let $\overline{B}_n$ denote the board $[-\frac{n-1}{2}, \frac{n-1}{2}] \times [-\frac{n-1}{2}, \frac{n-1}{2}]$.[4] A square of $\overline{B}_n$ will be known by the coordinates (x, y) of its corresponding vertex. Since we take up the queens version of the problem, we restrict our lines to having one of the following slopes: $0, +1, -1$, or ∞. These lines contain vertices of the lattice and when we write *line* we mean a line of this type. A subset S of the infinite square lattice will be called a *placement of queens* where there is a queen on each corresponding square of the chessboard. The *size* of a placement S is given by $|S|$. Two queens of a placement $\mathcal{Q}$ *define* a line if they lie on the same row, column or diagonal. The set of all collinear pairs of queens of $\mathcal{Q}$ are the set of lines that $\mathcal{Q}$ *defines*. A line is said to *cover* a square if the coordinates of the square is in the zero locus of the line. A placement is called *good* if it is maximal with respect

[4] This is the key difference to the set-up as compared to that of [3] as we now are considering a zero-sum grid, thus facilitating the use of Theorem 4.

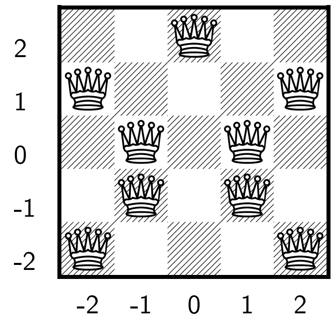

Figure 2: An example of a good placement with one lonely queen on a 5×5 board.

to the property of ‘no-3-in-a-line’; that is, a good placement does not have 3 collinear queens and any proper super-set of it does. A *lonely* queen is one that is not collinear with any other queen. We illustrate a placement with a lonely queen in Figure 2, where there exists one lonely queen in the topmost row (i.e., she has coordinates $(0, 2)$).

We now sketch the proof of Theorem 2. We will consider a good placement $\mathcal{Q}$ of size at most $4k + 1$. The proof will divide into two cases, which roughly depend on the number of lonely queens. In the first case, when the number of lonely queens is not one, when $q < 4k+1$ or when the number of lines defined by $\mathcal{Q}$ is not maximized, $\mathcal{Q}$ will define a set of lines and together these lines (and perhaps some additional lines) will be used to construct a polynomial that vanishes on the squares of the chessboard. However, we will be able to show that the coefficient on a leading monomial of this polynomial, which is of ‘small degree,’ is non-zero, thus obtaining a contradiction to Alon’s Combinatorial Nullstellensatz. (The proof in the first case is essentially that found in [3] for the proof of Theorem 1.) The second case, when the number of lonely queens is one, $q = 4k + 1$ and the number of lines defined by $\mathcal{Q}$ is maximized, is more involved. In this case, Alon’s Nullstellensatz *will be insufficient*. Like in the previous case, we use the set of lines defined by $\mathcal{Q}$ to construct several polynomials. For each of the four possible slopes of a line passing through the lonely queen, we define a polynomial so that it vanishes on all the squares of the chessboard. For each polynomial, we calculate the coefficient of an appropriate monomial, which is in terms of the coordinates of the lonely queen and sums of the intercepts of the defining lines of $\mathcal{Q}$. To compute this coefficient requires nothing more sophisticated than the Binomial Theorem. If one of these four coefficients is non-zero, then we obtain a contradiction to Nica’s Combinatorial Nullstellensatz for Zero-sum Grids. If each of these four coefficients is zero, then, together with some geometric and combinatorial arguments that yield additional equations, we will be able to construct a homogeneous system of linear equations whose solution yields the location of the lonely queen to be centered on the board (i.e., she has coordinates $(0, 0)$) along with some other attributes of the placement. This structural information allows for a combinatorial argument that finishes the proof.

3 Proof of Main Theorem

Proof of Theorem 2

Let $n = 4k + 1$ and $k \geq 1$. Let $\mathcal{Q}$ be a good placement on $\overline{B}_n$ with size $q = |\mathcal{Q}| \leq 4k + 1$. Let $\mathcal{Q}'$ denote the (possibly empty) subset of lonely queens in $\mathcal{Q}$. Let $|\mathcal{Q}'| = q'$. We work towards a contradiction.

Case 1: $q' \neq 1$, $q < 4k + 1$, or $q' = 1$ and some queen(s) that is not lonely participates in defining fewer than 4 lines.

We will give a polynomial $f(x, y)$ of total degree $8k$ so that $f(x, y) = 0$ for each square $(x, y) \in \overline{B}_n$. This will lead to a contradiction of the Combinatorial Nullstellensatz.[5]

We will define f as a product of linear factors. We first consider the set of lines defined by $\mathcal{Q}$. Since the placement $\mathcal{Q}$ is good, every unoccupied square of $\overline{B}_n$ is in the zero locus of at least one of these lines. For each lonely queen $Q_i \in \mathcal{Q}'$ we will 'artificially' define a new line that covers the square occupied by Q_i. Such an artificial line will receive a slope of either $\pm 1, 0$ or ∞. We make the choice so as to distribute the slopes as evenly as possible. Every occupied square is in the zero locus of at least one artificial line or one line defined by $\mathcal{Q}$.

For each of the four possible slopes there are at most $\left\lfloor \frac{q-q'}{2} \right\rfloor$ lines of that slope defined by $\mathcal{Q}$ and at most $\left\lceil \frac{q'}{4} \right\rceil$ artificial lines of that slope. If $q' \neq 1$ or if $q < 4k+1$, then $\left\lfloor \frac{q-q'}{2} \right\rfloor + \left\lceil \frac{q'}{4} \right\rceil \leq 2k$. If $q' = 1$ and some queen(s) that is not lonely participates in defining fewer than 4 lines, then one of the four possible slope directions has at most $2k - 1$ lines and we define a new line through the lonely queen to have that slope. If there are fewer than $2k$ lines of a particular slope, we add new lines so that there are $2k$ lines in total for each possible slope as this will facilitate an application of Theorem 3.

Let $\mathcal{L} = \{L_1, \ldots, L_{8k}\}$ be the set of $8k$ lines and let $l_i = 0$ be the equation in variables x and y defining L_i. We then define

$$f(x, y) = \prod_{i=1}^{8k} l_i \in \mathbb{R}[x, y].$$

Notice that the polynomial $f(x, y) = 0$ for every $(x, y) \in \overline{B}_n$ as every unoccupied square is covered by a line that $\mathcal{Q}$ defines and every occupied square is covered by an artificial line or a line defined by $\mathcal{Q}$. As $f(x, y)$ is the product of $8k$ linear factors, the total degree of $f(x, y)$ is

[5] This idea is originally to be found in [3] and it is in the latter case that new ideas are presented.

$8k$. By grouping the factors in f according to slope, we may express f as

$$f(x,y) = \prod_{j=1}^{2k} (x-\alpha_j)(y-\beta_j)(x-y-\gamma_j)(x+y-\delta_j)$$

for suitable constants $\alpha_j, \beta_j, \gamma_j, \delta_j$. The leading monomials of $f(x,y)$ are the same as that of $g(x,y) = x^{2k}y^{2k}(x^2-y^2)^{2k}$. The Binomial Theorem yields the coefficient of the top-degree term $x^{4k}y^{4k}$ to be $\pm\binom{2k}{k}$ – this coefficient is nonzero.

By applying Theorem 3 to $f(x,y)$, where $t_1 = t_2 = 4k$ and $S_1 = S_2 = \{-2k, \ldots, 2k\}$, we find $s_1 \in S_1, s_2 \in S_2$ for which $f(s_1, s_2) \neq 0$, a contradiction.

Case 2: $q = 4k+1$, $q' = 1$ and all other queens participate in defining a line of each of the four possible slopes.

As we did in Case 1, we employ the polynomial method; this time we will apply Theorem 4.

We start by constructing a polynomial $f_1 := f_1(x,y)$ of total degree $8k+1$ that vanishes on each square $(x,y) \in \overline{B}_n$. The polynomial f_1 will be a product of linear factors. We will take all linear factors that arise from the set of lines defined by $\mathcal{Q}$. Since the placement $\mathcal{Q}$ is good, every unoccupied square of $\overline{B}_n$ is in the zero locus of at least one of the lines defined by $\mathcal{Q}$. As given by the restrictions of the case, there is one lonely queen in $\mathcal{Q}'$ which is not on any defining line. For this one lonely queen $Q \in \mathcal{Q}'$ we artificially define a new line that passes through the square occupied by Q. While we are free to choose any one of the four possible slopes for this one line, at this point in the proof we choose slope ∞. Every occupied square is covered by at least one of these lines.

For each of the four possible slopes there are exactly $\frac{4k}{2} = 2k$ lines of that slope defined by $\mathcal{Q}$ and precisely one artificial line of slope ∞. (Unlike in Case 1, we never need to define any further lines.)

Let $\mathcal{L} = \{L_1, \ldots, L_{8k+1}\}$ be our set of $8k+1$ lines and let $l_i = 0$ be the equation in variables x and y defining L_i. We then define

$$f_1(x,y) = \prod_{i=1}^{8k+1} l_i \in \mathbb{R}[x,y].$$

Note that the polynomial $f_1(x,y) = 0$ for every $(x,y) \in \overline{B}_n$ as every unoccupied square is on a line defined by $\mathcal{Q}$ and every occupied square is on an artificial line or a line defined by $\mathcal{Q}$. By construction, the total degree of f is $8k+1$. Let (α_0, β_0) denote the square occupied by the lonely queen. By grouping the factors in f_1 according to slope, we may express f_1 as

$$f_1(x,y) = (x-\alpha_0)\prod_{j=1}^{2k} (x-\alpha_j)(y-\beta_j)(x-y-\gamma_j)(x+y-\delta_j)$$

for suitable constants $\alpha_j, \beta_j, \gamma_j, \delta_j$.

If can we conclude that the coefficient of the term $x^{4k}y^{4k}$ is nonzero, then we may apply Theorem 4 to $f_1(x, y)$, where $t_1 = t_2 = 4k$ and $S_1 = S_2 = \{-2k, \ldots, 2k\}$ (zero-sum sets), to obtain that there are $s_1 \in S_1, s_2 \in S_2$ such that $f_1(s_1, s_2) \neq 0$, thus obtaining a contradiction.

We thus 'expand' $f_1(x, y)$ and 'collect' like terms so that we might determine the coefficient of $x^{4k}y^{4k}$. Since we are only interested in the coefficient on this term, we focus our analysis only on it. Note that this term has degree one less than the degree of f_1. So, to obtain such a monomial in the expansion, from the $8k + 1$ linear factors, we must choose the x-variable $4k$ times, the y-variable $4k$ times and thus some constant once. We think of choosing that constant first and so partition our analysis based upon whether the constant that we have chosen is some $\alpha_j, \beta_j, \gamma_j$ or δ_j.

1. Choose some α_j first for some $0 \leq j \leq 2k$.

   The remaining factors with their constant terms removed (since we can't choose them) are:
$$x^{2k}y^{2k}(x - y)^{2k}(x + y)^{2k}.$$
   This equals
$$x^{2k}y^{2k}(x^2 - y^2)^{2k} = x^{2k}y^{2k}\sum_{\ell=0}^{2k}\binom{2k}{\ell}(x^2)^{\ell}(-y^2)^{2k-\ell}.$$
   The only choice of ℓ which yields the desired monomial is $\ell = k$, which gives a coefficient of $\binom{2k}{k}(-1)^k$. This is the contribution for each α_j. Thus, we have a total contribution to the coefficient of the desired monomial of $(-1)^k\binom{2k}{k}\sum_{j=0}^{2k} -\alpha_j$.

2. Choose some β_j first for some $1 \leq j \leq 2k$.

   The remaining factors with their constant terms removed (since we can't choose them) are:
$$x^{2k+1}y^{2k-1}(x - y)^{2k}(x + y)^{2k}.$$
   This equals
$$x^{2k+1}y^{2k-1}(x^2 - y^2)^{2k} = x^{2k+1}y^{2k-1}\sum_{\ell=0}^{2k}\binom{2k}{\ell}(x^2)^{\ell}(-y^2)^{2k-\ell}.$$
   There is no choice of ℓ which yields the desired monomial, which is true for each β_j. Thus, we have a total contribution to the coefficient of the desired monomial of 0.

3. Choose some γ_j first for some $1 \leq j \leq 2k$.

The remaining factors with their constant terms removed (since we can't choose them) are:

$$x^{2k+1}y^{2k}(x-y)^{2k-1}(x+y)^{2k}.$$

This equals

$$\begin{aligned}
x^{2k+1}y^{2k}(x+y)(x^2-y^2)^{2k-1} &= x^{2k+1}y^{2k}(x+y)\sum_{\ell=0}^{2k-1}\binom{2k-1}{\ell}(x^2)^{\ell}(-y^2)^{2k-1-\ell} \\
&= x^{2k+2}y^{2k}\sum_{\ell=0}^{2k-1}\binom{2k-1}{\ell}(x^2)^{\ell}(-y^2)^{2k-1-\ell} \\
&\quad +x^{2k+1}y^{2k+1}\sum_{\ell=0}^{2k-1}\binom{2k-1}{\ell}(x^2)^{\ell}(-y^2)^{2k-1-\ell}.
\end{aligned}$$

Only in the first of the two summands is there a choice of ℓ which yields the desired monomial; it is $\ell = k-1$, which gives a coefficient of $\binom{2k-1}{k-1}(-1)^k$. This is the contribution for each γ_j. Thus, we have a total contribution to the coefficient of the desired monomial of $(-1)^k\binom{2k-1}{k-1}\sum_{j=1}^{2k}-\gamma_j$.

4. Choose some δ_j first for some $1 \leq j \leq 2k$.

The remaining factors with their constant terms removed (since we can't choose them) are:

$$x^{2k+1}y^{2k}(x-y)^{2k}(x+y)^{2k-1}.$$

This equals

$$\begin{aligned}
x^{2k+1}y^{2k}(x-y)(x^2-y^2)^{2k-1} &= x^{2k+1}y^{2k}(x-y)\sum_{\ell=0}^{2k-1}\binom{2k-1}{\ell}(x^2)^{\ell}(-y^2)^{2k-1-\ell} \\
&= x^{2k+2}y^{2k}\sum_{\ell=0}^{2k-1}\binom{2k-1}{\ell}(x^2)^{\ell}(-y^2)^{2k-1-\ell} \\
&\quad -x^{2k+1}y^{2k+1}\sum_{\ell=0}^{2k-1}\binom{2k-1}{\ell}(x^2)^{\ell}(-y^2)^{2k-1-\ell}.
\end{aligned}$$

Only in the first of the two summands is there a choice of ℓ which yields the desired monomial; it is $\ell = k-1$, which gives a coefficient of $\binom{2k-1}{k-1}(-1)^k$. This is the contribution for each δ_j. Thus, we have a total contribution to the coefficient of the desired monomial of $(-1)^k\binom{2k-1}{k-1}\sum_{j=1}^{2k}-\delta_j$.

The sum of these four contributions is the coefficient $C^{f_1}(4k,4k)$ on $x^{4k}y^{4k}$ in $f_1(x,y)$. It is

$$C^{f_1}(4k,4k) \quad = \quad (-1)^k\binom{2k}{k}\sum_{j=0}^{2k}-\alpha_j + (-1)^k\binom{2k-1}{k-1}\sum_{j=1}^{2k}-\gamma_j + (-1)^k\binom{2k-1}{k-1}\sum_{j=1}^{2k}-\delta_j.$$

If this coefficient is non-zero, then by Theorem 4 we are done. Thus, assume it is zero.

We repeat the above procedure for the placement $\mathcal{Q}$ with the one lonely queen at (α_0, β_0) by defining a similar polynomial $f_2(x, y)$, this time with a line of slope 0 through the lonely queen, as follows.

Let

$$f_2(x,y) = (y - \beta_0) \prod_{j=1}^{2k} (x - \alpha_j)(y - \beta_j)(x - y - \gamma_j)(x + y - \delta_j)$$

for suitable constants $\alpha_j, \beta_j, \gamma_j, \delta_j$.

By repeating the above procedure or, perhaps more simply, by noting the symmetries between $f_1(x, y)$ and $f_2(x, y)$, the coefficient $C^{f_2}(4k, 4k)$ on $x^{4k}y^{4k}$ in $f_2(x, y)$ is

$$C^{f_2}(4k,4k) \quad = \quad (-1)^k \binom{2k}{k} \sum_{j=0}^{2k} -\beta_j + (-1)^k \binom{2k-1}{k-1} \sum_{j=1}^{2k} -\gamma_j + (-1)^k \binom{2k-1}{k-1} \sum_{j=1}^{2k} -\delta_j.$$

If this coefficient is non-zero, then by Theorem 4 we are done. Thus, assume it is zero.

We repeat the above procedure for the placement $\mathcal{Q}$ with the one lonely queen at (α_0, β_0) by defining a similar polynomial $f_3(x, y)$, this time with a line of slope $+1$ through the lonely queen, as follows.

Let

$$f_3(x,y) = (x - y - \gamma_0) \prod_{j=1}^{2k} (x - \alpha_j)(y - \beta_j)(x - y - \gamma_j)(x + y - \delta_j)$$

for suitable constants $\alpha_j, \beta_j, \gamma_j, \delta_j$.

As above, we 'expand' $f_3(x, y)$ and 'collect' like terms so that we might determine the coefficient of $x^{4k}y^{4k}$. Since we are only interested in the coefficient on this term, we focus our analysis only on it. Note that this term has degree one less than the degree of f_3. So, to obtain such a monomial in the expansion, from the $8k + 1$ linear factors, we must choose the x-variable $4k$ times, the y-variable $4k$ times and thus some constant once. We think of choosing that constant first and so partition our analysis based upon whether the constant that we have chosen is some $\alpha_j, \beta_j, \gamma_j$ or δ_j.

1. Choose some α_j first for some $1 \leq j \leq 2k$.
   The remaining factors with their constant terms removed (since we can't choose them) are:
   $$x^{2k-1}y^{2k}(x - y)^{2k+1}(x + y)^{2k}.$$

This equals

$$
\begin{aligned}
x^{2k-1}y^{2k}(x-y)(x^2-y^2)^{2k} &= x^{2k-1}y^{2k}(x-y)\sum_{\ell=0}^{2k}\binom{2k}{\ell}(x^2)^{\ell}(-y^2)^{2k-\ell} \\
&= x^{2k}y^{2k}\sum_{\ell=0}^{2k}\binom{2k}{\ell}(x^2)^{\ell}(-y^2)^{2k-\ell} \\
&\quad -x^{2k-1}y^{2k+1}\sum_{\ell=0}^{2k}\binom{2k}{\ell}(x^2)^{\ell}(-y^2)^{2k-\ell}.
\end{aligned}
$$

Only in the first of the two summands is there a choice of ℓ which yields the desired monomial; it is $\ell = k$, which gives a coefficient of $\binom{2k}{k}(-1)^k$. This is the contribution for each α_j. Thus, we have a total contribution to the coefficient of the desired monomial of $(-1)^k\binom{2k}{k}\sum_{j=1}^{2k} -\alpha_j$.

2. Choose some β_j first for some $1 \le j \le 2k$.

   The remaining factors with their constant terms removed (since we can't choose them) are:

$$x^{2k}y^{2k-1}(x-y)^{2k+1}(x+y)^{2k}.$$

   This equals

$$
\begin{aligned}
x^{2k}y^{2k-1}(x-y)(x^2-y^2)^{2k} &= x^{2k}y^{2k-1}(x-y)\sum_{\ell=0}^{2k}\binom{2k}{\ell}(x^2)^{\ell}(-y^2)^{2k-\ell} \\
&= x^{2k+1}y^{2k-1}\sum_{\ell=0}^{2k}\binom{2k}{\ell}(x^2)^{\ell}(-y^2)^{2k-\ell} \\
&\quad -x^{2k}y^{2k}\sum_{\ell=0}^{2k}\binom{2k}{\ell}(x^2)^{\ell}(-y^2)^{2k-\ell}.
\end{aligned}
$$

   Only in the second of the two summands is there a choice of ℓ which yields the desired monomial; it is $\ell = k$, which gives a coefficient of $(-1)\binom{2k}{k}(-1)^k$. This is the contribution for each β_j. Thus, we have a total contribution to the coefficient of the desired monomial of $(-1)^{k+1}\binom{2k}{k}\sum_{j=1}^{2k} -\beta_j$.

3. Choose some γ_j first for some $0 \le j \le 2k$.

   The remaining factors with their constant terms removed (since we can't choose them) are:

$$x^{2k}y^{2k}(x-y)^{2k}(x+y)^{2k}.$$

This equals

$$x^{2k}y^{2k}(x^2-y^2)^{2k} \quad = \quad x^{2k}y^{2k}\sum_{\ell=0}^{2k}\binom{2k}{\ell}(x^2)^{\ell}(-y^2)^{2k-\ell}.$$

The only choice of ℓ which yields the desired monomial is $\ell = k$, which gives a coefficient of $\binom{2k}{k}(-1)^k$. This is the contribution for each γ_j. Thus, we have a total contribution to the coefficient of the desired monomial of $(-1)^k\binom{2k}{k}\sum_{j=0}^{2k}-\gamma_j$.

4. Choose some δ_j first for some $1 \le j \le 2k$.

The remaining factors with their constant terms removed (since we can't choose them) are:

$$x^{2k}y^{2k}(x-y)^{2k+1}(x+y)^{2k-1}.$$

This equals

$$\begin{aligned}
x^{2k}y^{2k}(x-y)^2(x^2-y^2)^{2k-1} \quad &= \quad x^{2k}y^{2k}(x^2-2xy+y^2)\sum_{\ell=0}^{2k-1}\binom{2k-1}{\ell}(x^2)^{\ell}(-y^2)^{2k-1-\ell} \\
&= \quad x^{2k+2}y^{2k}\sum_{\ell=0}^{2k-1}\binom{2k-1}{\ell}(x^2)^{\ell}(-y^2)^{2k-1-\ell} \\
&\quad\quad -2x^{2k+1}y^{2k+1}\sum_{\ell=0}^{2k-1}\binom{2k-1}{\ell}(x^2)^{\ell}(-y^2)^{2k-1-\ell} \\
&\quad\quad +x^{2k}y^{2k+2}\sum_{\ell=0}^{2k-1}\binom{2k-1}{\ell}(x^2)^{\ell}(-y^2)^{2k-1-\ell}.
\end{aligned}$$

Amongst the three summands, there is a choice of $\ell = k-1$ in the first, no choice in the second and a choice of $\ell = k$ in the third. This yields a coefficient of $(-1)^k\binom{2k-1}{k-1} + (-1)^{k-1}\binom{2k-1}{k} = 0$. This is the contribution for each δ_j. Thus, we have a total contribution to the coefficient of the desired monomial of 0.

The sum of these four contributions is the coefficient $C^{f_3}(4k,4k)$ on $x^{4k}y^{4k}$ in $f_3(x,y)$. It is

$$C^{f_3}(4k,4k) \quad = \quad (-1)^k\binom{2k}{k}\sum_{j=1}^{2k}-\alpha_j + (-1)^{k+1}\binom{2k}{k}\sum_{j=1}^{2k}-\beta_j + (-1)^k\binom{2k}{k}\sum_{j=0}^{2k}-\gamma_j.$$

If this coefficient is non-zero, then by Theorem 4 we are done. Thus, assume it is zero.

We repeat the above procedure for the placement $\mathcal{Q}$ with the one lonely queen at (α_0, β_0) by defining a similar polynomial $f_4(x,y)$, this time with a line of slope -1 through the lonely queen, as follows.

Let

$$f_4(x,y) = (x+y-\delta_0)\prod_{j=1}^{2k}(x-\alpha_j)(y-\beta_j)(x-y-\gamma_j)(x+y-\delta_j)$$

for suitable constants $\alpha_j, \beta_j, \gamma_j, \delta_j$.

By repeating the above procedure or, perhaps more simply, by noting the symmetries between $f_3(x,y)$ and $f_4(x,y)$, the coefficient $C^{f_4}(4k,4k)$ on $x^{4k}y^{4k}$ in $f_4(x,y)$ is

$$C^{f_4}(4k,4k) \quad = \quad (-1)^k\binom{2k}{k}\sum_{j=1}^{2k}-\alpha_j + (-1)^k\binom{2k}{k}\sum_{j=1}^{2k}-\beta_j + (-1)^k\binom{2k}{k}\sum_{j=0}^{2k}-\delta_j.$$

If this coefficient is non-zero, then by Theorem 4 we are done. Thus, assume it is zero.

We have now reached the following system of linear equations:

$$C^{f_1}(4k,4k) = 0, C^{f_2}(4k,4k) = 0, C^{f_3}(4k,4k) = 0, C^{f_4}(4k,4k) = 0. \tag{1}$$

Next we generate four additional linear equations via some geometric and combinatorial observations.

Consider the lonely queen located on square (α_0, β_0): the values of α_0 and β_0 determine the values of γ_0 and δ_0 as follows. The line of slope $+1$ that goes through the square (α_0, β_0) has equation

$$y - \beta_0 = 1(x-\alpha_0), \quad x - y - (\alpha_0 - \beta_0) = 0$$

and the line of slope -1 that goes through the square (α_0, β_0) has equation

$$y - \beta_0 = -1(x-\alpha_0), \quad x + y - (\alpha_0 + \beta_0) = 0.$$

As a result, we have

$$\alpha_0 - \beta_0 - \gamma_0 = 0, \tag{2}$$

$$\alpha_0 + \beta_0 - \delta_0 = 0. \tag{3}$$

Now consider the other $4k$ queens of $\mathcal{Q}\setminus\mathcal{Q}'$ (i.e., those that are not lonely). For each such queen there exists an $\alpha \in \{\alpha_1, \ldots, \alpha_{2k}\}$ and $\beta \in \{\beta_1, \ldots, \beta_{2k}\}$ that give her coordinates. The line of slope $+1$ that goes through the square (α, β) has equation

$$y - \beta = 1(x-\alpha), \quad x - y - (\alpha - \beta) = 0$$

and the line of slope -1 that goes through the square (α, β) has equation

$$y - \beta = -1(x-\alpha), \quad x + y - (\alpha + \beta) = 0.$$

As a result, we have $\gamma = \alpha - \beta$ for some $\gamma \in \{\gamma_1, \ldots, \gamma_{2k}\}$ and $\delta = \alpha + \beta$ for some $\delta \in \{\delta_1, \ldots, \delta_{2k}\}$. As each such diagonal line is defined by two queens, upon considering the $4k$ equations deriving from the -1-slope lines each element of $\{\gamma_1, \ldots, \gamma_{2k}\}$ occurs twice in this set of equations; similarly, in the $4k$ equations deriving from the $+1$-slope lines each element of $\{\delta_1, \ldots, \delta_{2k}\}$ occurs twice. Thus, we can write the following:

$$\sum_{Q \in \mathcal{Q} \setminus \mathcal{Q}'} (\alpha - \beta) = 2 \sum_{i=1}^{2k} \gamma_i, \tag{4}$$

and

$$\sum_{Q \in \mathcal{Q} \setminus \mathcal{Q}'} (\alpha + \beta) = 2 \sum_{i=1}^{2k} \delta_i. \tag{5}$$

The restrictions of Case 2 give that each queen in $\mathcal{Q} \setminus \mathcal{Q}'$ is contained in both a vertical line and a horizontal line. As a result, each $\alpha \in \{\alpha_1, \ldots, \alpha_{2k}\}$ and each $\beta \in \{\beta_1, \ldots, \beta_{2k}\}$ appears twice on the left side of each of Equation 4 and Equation 5. This enables us to rewrite the left side of Equations 4 and 5 to obtain

$$2 \sum_{i=1}^{2k} \alpha_i - 2 \sum_{i=1}^{2k} \beta_i = 2 \sum_{i=1}^{2k} \gamma_i, \tag{6}$$

and

$$2 \sum_{i=1}^{2k} \alpha_i + 2 \sum_{i=1}^{2k} \beta_i = 2 \sum_{i=1}^{2k} \delta_i. \tag{7}$$

We may scale and rewrite Equations 6-7 as

$$\sum_{i=1}^{2k} \alpha_i - \sum_{i=1}^{2k} \beta_i - \sum_{i=1}^{2k} \gamma_i = 0, \tag{8}$$

and

$$\sum_{i=1}^{2k} \alpha_i + \sum_{i=1}^{2k} \beta_i - \sum_{i=1}^{2k} \delta_i = 0. \tag{9}$$

At this point, we have now generated a homogeneous system of eight linear equations - these are Equations 1, 2, 3, 8, 9 - in the variables $\alpha_0, \beta_0, \gamma_0, \delta_0, \sum_{j=1}^{2k} \alpha_j, \sum_{j=1}^{2k} \beta_j, \sum_{j=1}^{2k} \gamma_j, \sum_{j=1}^{2k} \delta_j$. For notational convenience we set $\omega := (-1)^k \binom{2k}{k}$ and express these equations using the following augmented matrix.

$$
[\mathbf{A} \;\; \mathbf{0}] = \begin{array}{c}
\begin{array}{cccccccc} \alpha_0 & \beta_0 & \gamma_0 & \delta_0 & \sum_{j=1}^{2k} \alpha_j & \sum_{j=1}^{2k} \beta_j & \sum_{j=1}^{2k} \gamma_j & \sum_{j=1}^{2k} \delta_j \end{array} \\
\left[\begin{array}{ccccccccc}
\omega & 0 & 0 & 0 & \omega & 0 & \omega/2 & \omega/2 & 0 \\
0 & \omega & 0 & 0 & 0 & \omega & \omega/2 & \omega/2 & 0 \\
0 & 0 & \omega & 0 & \omega & -\omega & \omega & 0 & 0 \\
0 & 0 & 0 & \omega & \omega & \omega & 0 & \omega & 0 \\
1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0
\end{array}\right]
\end{array}
$$

Guassian elimination yields the following row reduced 8×8 coefficient matrix:

$$
\mathbf{A} \sim \begin{array}{c}
\begin{array}{cccccccc} \alpha_0 & \beta_0 & \gamma_0 & \delta_0 & \sum_{j=1}^{2k} \alpha_j & \sum_{j=1}^{2k} \beta_j & \sum_{j=1}^{2k} \gamma_j & \sum_{j=1}^{2k} \delta_j \end{array} \\
\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -0.5 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -0.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
$$

We see that the null space of A is spanned by the following vector,

$$
\begin{bmatrix} 1 & 1 & 0 & 2 & -0.5 & -0.5 & 0 & -1 \end{bmatrix}^{\intercal}.
$$

Note, for a vector $\mathbf{x}$ *not* in the null space, at least one entry of $A\mathbf{x}$ is nonzero. If this nonzero entry is among the first 4 entries, then at least one of $C^{f_1}(4k,4k), C^{f_2}(4k,4k), C^{f_3}(4k,4k), C^{f_4}(4k,4k)$ is nonzero, a contradiction to Theorem 4. If this nonzero entry is among the latter 4 entries, then it contradicts a relationship we found in one of Equations 2, 3, 8, 9.

The vectors in the null space of A imply that the lonely queen has coordinates (s,s) for $s \in \{-2k,\dots,2k\}$, so the lonely queen is on the line $y = x$, i.e. the 'back-diagonal' of the chessboard. If we rotate the placement by 90-degrees counter-clockwise, then the placement we obtain satisfies the conditions of Case 2 and has the lonely queen on the 'forward-diagonal', i.e., on

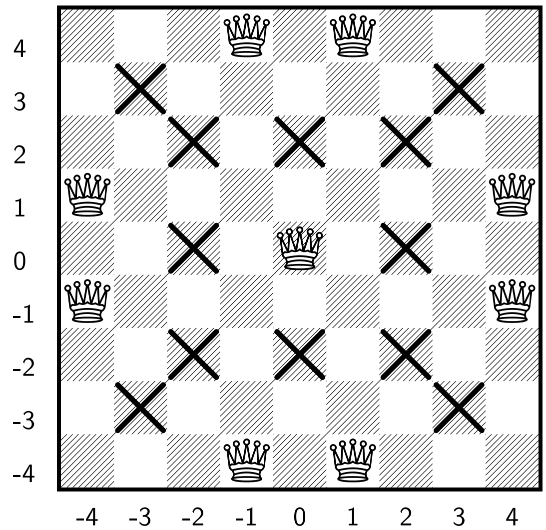

Figure 3: A placement of 10 queens on a 9×9 board which corresponds to the zero vector in Nul A. Sqaures marked with a cross indicate those squares for which the additional placement of a queen would *not* yield three-in-a-line.

the line $y = -x$. Applying all of the previous arguments to this new placement shows that the lonely queen must be on the back-diagonal. Thus, the lonely queen is at the center, i.e., she has coordinates $(0, 0)$. This implies that the only vector $\mathbf{x}$ in the null space of A which we need to be concerned with is the zero vector. Thus,

$$\alpha_0, \beta_0, \gamma_0, \delta_0, \sum_{j=1}^{2k} \alpha_j, \sum_{j=1}^{2k} \beta_j, \sum_{j=1}^{2k} \gamma_j, \sum_{j=1}^{2k} \delta_j = 0. \tag{10}$$

With the location information provided by Equation 10 and the restrictions of Case 2, we now turn to a combinatorial and geometric argument to finish the proof.[6]

Let $U \subseteq \overline{B}_n$ be the set of squares left uncovered by those defining lines of $\mathcal{Q}$ of slope 0 or ∞ (and recall that the lonely queen *does not* define a line). Notice that U defines a rectangular sub-board. For any index $i \in \{\frac{-(n-1)}{2}, \ldots, \frac{n-1}{2}\}$ (respectively $j \in \{\frac{-(n-1)}{2}, \ldots, \frac{n-1}{2}\}$) let $C_i = \{(i, \ell) \in U : \frac{-(n-1)}{2} \leq \ell \leq \frac{n-1}{2}\}$ (respectively $R_j = \{(\ell, j) \in U : \frac{-(n-1)}{2} \leq \ell \leq \frac{n-1}{2}\}$). That is, C_i is the set of squares of U in column i and R_j is the set of squares of U in row j. Let $a < b$ be the minimum and maximum indices, respectively, for which $C_i \neq \emptyset$. Define $a' < b'$ analogously for the sets R_j. The number of the C_i and R_j that are nonempty is $n - 2k = 4k + 1 - 2k = 2k + 1$. The $8k$ squares that form the set $C_a \cup C_b \cup R_{a'} \cup R_{b'}$ will be referred to as the *perimeter of* U.

[6]Figure 3 is an example of a placement satisfying Equation 10 yet is not good.

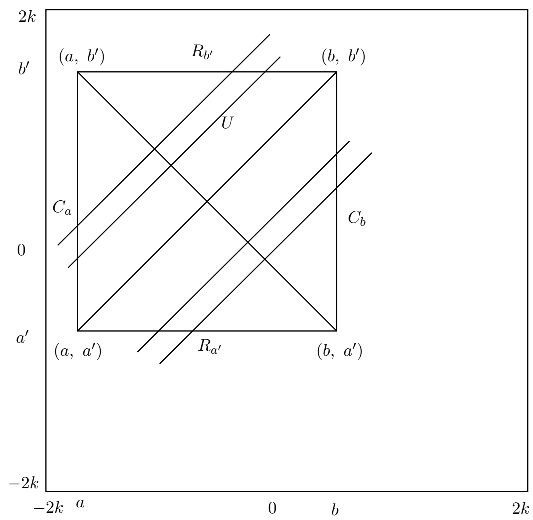

Figure 4: The $8k$ squares that form the perimeter of U and some covering lines.

Without loss of generality, we may assume $b - a \geq b' - a'$ as otherwise we may rotate the placement by 90°; also, note that $b - a \geq b' - a' \geq 2k$. Consider the case that $b - a > b' - a'$ (i.e. the inequality is strict, U is rectangular but not square). In this case, there are at least $4k + 1$ (and at most $4k + 2$) empty squares of $C_a \cup C_b$, each of which must be covered by a line of slope ± 1. However, each such line can cover at most one of these squares. As there are $4k$ such lines, we fall short of being able to cover each such square. So, we may conclude that $b - a = b' - a'$.

If the lonely queen were to occupy a square in C_a (that is, $a = 0$), then all vertical lines defined by $\mathcal{Q}$ would be to the left of center contradicting that $\sum_{i=1}^{2k} \alpha_i = 0$. A similar contradiction would be reached if the lonely queen were to occupy a square in C_b (that is, $b = 0$): all vertical lines defined by $\mathcal{Q}$ would be to the right of center contradicting that $\sum_{i=1}^{2k} \alpha_i = 0$. Thus, the squares of $C_a \cup C_b$ are empty of queens. Likewise, the squares of $R_{a'} \cup R_{b'}$ are empty of queens. Thus, the perimeter of U is empty of queens.

Consider the $8k$ squares that form the perimeter of U as shown in Figure 4. As the perimeter of U is empty of queens, each of these squares must be covered by a line of slope ± 1. Such a line can cover at most 2 such squares. With a total of $4k$ diagonal lines ($2k$ of slope $+1$ and $2k$ of slope -1) and $8k$ squares, each diagonal line must cover 2 squares. This forces the -1-slope diagonal covering the squares (a, b') and (b, a') and forces the $+1$-slope diagonal covering the squares (a, a') and (b, b') to exist. Also, a -1-slope line that covers a square of $C_a \setminus \{(a, a'), (a, b')\}$ must also cover a square of $R_{a'} \setminus \{(a, a'), (b, a')\}$; likewise, a -1-slope line that covers a square of $C_b \setminus \{(b, a'), (b, b')\}$ must also cover a square of $R_{b'} \setminus \{(a, b'), (b, b')\}$. Also, a $+1$-slope line that covers a square of $C_a \setminus \{(a, a'), (a, b')\}$ must also cover a square of $R_{b'} \setminus \{(a, b'), (b, b')\}$; likewise, a $+1$-slope line that covers a square of $C_b \setminus \{(b, a'), (b, b')\}$

must also cover a square of $R_{a'} \setminus \{(a,a'),(b,a')\}$. Suppose that there are p lines of slope -1 covering squares of $(C_a \setminus \{(a,a'),(a,b')\}) \cup (R_{a'} \setminus \{(a,a'),(b,a')\})$. The remaining $2k-1-p$ lines of slope -1 cover the squares of $(C_b \setminus \{(b,a'),(b,b')\}) \cup (R_{b'} \setminus \{(a,b'),(b,b')\})$, leaving $4k-2-2(2k-1-p) = 2p$ squares of this set uncovered, with p squares in $C_b \setminus \{(b,a'),(b,b')\}$ and p squares in $R_{b'} \setminus \{(a,b'),(b,b')\}$. These squares must be covered by $+1$-slope lines, for an additional $2p$ lines of slope $+1$. These $2p$ lines together with the $+1$-slope diagonal covering (a,a') and (b,b') give a total of $2p+1$ such lines, which is an odd number of such lines. However, we said that the number of lines of slope $+1$ is $2k$, which is even, and this contradiction finishes the proof.

4 Concluding remarks

4.1 Placements that correspond to the zero vector

When $n = 8k+1$ we can describe some placements that correspond to the zero vector in Nul A as follows. Select k squares with coordinates $(x_1,y_1),\ldots,(x_k,y_k)$ such that these meet the following conditions: $0 < x_i, y_i \le 2k$, $\frac{y_i}{x_i} < 1$, the values $x_1,y_1,\ldots,x_k,y_k$ are distinct and for any $i \ne j$ we have $\frac{y_i}{x_i} \ne \frac{y_j}{x_j}$. Next consider 'folding' the board across the $x-$axis, then folding it across the $y-$axis, and finally folding it along the line $y = x$. Consider the 8 squares that 'stack' on top of a selected square and place a queen in each of these. Do this for each selected square. We have now placed $8k$ queens. Finally, place a lonely queen at $(0,0)$, for a total of $8k+1$ queens. It is easy to check that this placement is in Nul A. From a geometric perspective, the placement will look like k octagons that are 'nested' and centered at $(0,0)$. The placement given in Figure 3 is such a placement derived from this scheme where $n = 9$ and the set of initial selected squares consists of one square with coordinates $(4,1)$; it is one of six possible placements for $n = 9$ via this scheme.

When $n = 5$ there are no placements that meet the conditions of Case 2 since a queen in $\mathcal{Q} \setminus \mathcal{Q}'$ forces the existence of four other queens and together with the lonely queen the placement would have at least $6 > n$ queens. When $n = 8k+5$ we have not found any placements that correspond to the zero vector in Nul A.

4.2 Other cases to consider

When $n = 4k+3$, we also believe that $m_3(n) \ge n+1$. At this point of our investigations, we have not been able to establish this lower bound. However, there are some sub-cases for which we can establish this improved lower bound: we sketch the proof here by following the polynomial method template provided in Case 1 of the proof of Theorem 2. By considering a good placement $\mathcal{Q}$ of size at most $4k+3$ on $\overline{B}_n$ with at most $2k$ defined lines of slope ± 1, the polynomial of degree $8k+4$

$$g(x,y) = \prod_{j=1}^{2k+2} (x-\alpha_j)(y-\beta_j) \prod_{j=1}^{2k} (x-y-\gamma_j)(x+y-\delta_j)$$

can be constructed so as to vanish on all squares of the chessboard yet has a non-zero coefficient on the monomial $x^{4k+2}y^{4k+2}$ by the Binomial Theorem. Thus, by Theorem 3 we reach a contradiction. This approach will settle all cases where $q' \geq 2$ and some others. However, one particularly challenging sub-case that we cannot resolve is when $q' = 0$ and the number of lines defined by $\mathcal{Q}$ is maximized (i.e. there are $2k+1$ lines defined by $\mathcal{Q}$ in each of the four slopes). In this sub-case, one might consider the following polynomial

$$h(x,y) = \prod_{j=1}^{2k+1} (x-\alpha_j)(y-\beta_j)(x-y-\gamma_j)(x+y-\delta_j).$$

A leading monomial suitable for an application of the Combinatorial Nullstellensatz is again $x^{4k+2}y^{4k+2}$. However, this monomial has the same coefficient as the same monomial in $x^{2k+1}y^{2k+1}(x^2-y^2)^{2k+1}$, which is zero. Thus, a polynomial method approach seems problematic. At the same time, the combinatorial approach that comes towards the end of Case 2 in the proof of Theorem 2 seems problematic for the following reason. In such a placement, there will be up to four queens that are neither lonely nor in a defining line of each possible slope. Gaining information about the coordinates of these queens appears difficult and it could be that these particular queens are located on the perimeter of U.

Turning to a different set of cases: there are small cases of n even where it has been established that $m_3(n) = n+1$. From Table 1, we see that these values are $n = 8, 14, 16, 20, 22, 24$. There is no discernible pattern to us.

Recently, Di Stefano, Klavžar, Krishnakumar, Tuite and Yero [4] have extended Gardner's problem to graph theory. This, too, looks like an interesting line of inquiry.

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