On the size and structure of graphs with a constant number of 1-factors

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Abstract

We investigate the maximum number of edges in a graph with a prescribed number of 1-factors. We also examine the structure of such extremal graphs.

Keywords: 1-factor, extremal graph

1 Introduction

Let G be a graph with n vertices. Throughout we assume that n is an even integer. A 1-*factor* of a graph G is a spanning 1-regular subgraph of G. Let $\Phi(G)$ denote the number of 1-factors in G.

The problem of determining the number of 1-factors in graphs with certain properties has been studied by several researchers (see, e.g., [1, 2, 3, 7]). Here we are interested in the maximum number of edges in a graph with a prescribed number of 1-factors.

We denote the complete graph on t vertices by K_t. We will extensively use the following graph operations. The *union* $G_1 \cup G_2$ of graphs G_1 and G_2 with disjoint vertex and edge sets is the graph with $V(G_1) \cup V(G_2)$ and $E(G_1) \cup E(G_2)$. The *join* of graphs G_1 and G_2, written $G_1 \vee G_2$, is the graph obtained from $G_1 \cup G_2$ by adding the edges $\{xy : x \in V(G_1), y \in V(G_2)\}$.

It was shown by Hetyei (cf. [6]) that the maximum number of edges in an n-vertex graph G with exactly one 1-factor (i.e. $\Phi(G) = 1$) is $\frac{n^2}{4}$. The n-vertex extremal graph H_n, that is the graph with exactly one 1-factor and $\frac{n^2}{4}$ edges, is unique. For $n = 2$ it is K_2 and for $n \geq 4$ we can define it recursively as $H_n = K_1 \vee (H_{n-2} \cup K_1)$. We refer to this result as *Hetyei's Theorem*. To simplify notation, we let $h(G)$ be the graph obtained from G by adding one vertex v adjacent to all vertices of G and then adding one more vertex u adjacent only to v.

The aim of this article is to present some results on the size and structure of graphs when $\Phi(G)$ is a fixed integer larger than 1. We denote the maximum number of edges in an n-vertex graph with

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precisely p 1-factors by $f(n,p)$; otherwise $f(n,p)=0$ if there is no n-vertex G with $\Phi(G)=p$. For example, Hetyei's Theorem says that $f(n,1)=\frac{n^2}{4}$ for all positive even n. Quite surprisingly, in general the function $f(n,p)$ is not monotonic in p (cf. Remark 2.6). This irregular behavior makes this function more interesting (and also more difficult) to examine.

In this article, we precisely determine all nonzero $f(n,p)$ for small values of p. Moreover and more importantly, we describe all extremal graphs with exactly two or three 1-factors.[1]

2 Graphs with a small number of 1-factors

We start with a simple lemma.

Lemma 2.1 *If $f(n,p)>0$, then $f(n+2,p)\geq f(n,p)+(n+1)$. Consequently, if $f(n,p)\geq \frac{n^2}{4}+c$, then $f(n+2,p)\geq \frac{(n+2)^2}{4}+c$.*

Proof. Let G_n be an extremal graph of order n with $\Phi(G_n)=p$. Define recursively $G_{n+2}=h(G_n)$. Note that $\Phi(G_{n+2})=\Phi(G_n)=p$. Hence,

$$f(n+2,p)\geq |E(G_{n+2})|=|E(G_n)|+(n+1)=f(n,p)+(n+1),$$

as required. □

Now we define an auxiliary family of graphs $\{F_4,F_6,\dots\}$. Let K_4-e denote the graph obtained from K_4 by deleting one edge. Let $F_4=K_4-e$ and denote by t and u the two vertices of K_4-e with degree 3. Let H_n be the unique extremal graph such that $\Phi(H_n)=1$ (cf. Introduction). For $n\geq 6$, let F_n be the graph $(K_4-e)\cup H_{n-4}$ together with all the edges joining each of t and u to all of $V(H_{n-4})$. Note that F_n is a graph of order n with

$$|E(F_n)|=5+\frac{(n-4)^2}{4}+2(n-4)=\frac{n^2}{4}+1$$

and $\Phi(F_n)=2$.

Now we determine the maximum number of edges in graphs with precisely two 1-factors. Moreover, we also describe all such graphs. Some ideas of the proof of the next theorem come from [5].

Theorem 2.2 *If n is even and $n\geq 4$, then $f(n,2)=\frac{n^2}{4}+1$; otherwise $f(n,2)=0$. Furthermore, for every $n\geq 4$ there are precisely $\frac{n-2}{2}$ extremal graphs defined recursively for $1\leq i\leq \frac{n-2}{2}$ as follows:*

$$G_n^i=\begin{cases} F_n & \text{for } i=\frac{n-2}{2},\\ h(G_{n-2}^i) & \text{for } 1\leq i\leq \frac{n-4}{2}.\end{cases}$$

Proof. As K_4-e has two 1-factors, we have $f(4,2)\geq\frac{4^2}{4}+1$, and so the lower bound follows from Lemma 2.1.

Now we show the upper bound. Let G be an n-vertex graph with $\Phi(G)=2$. Denote the edge sets of the two distinct 1-factors by R and B, viewed as red edges and blue edges, respectively,

[1] Recently, we learned that in [4] the authors refined some of our results. (The previous version of our manuscript was available online.)

with $R = \{r_1, r_2, \ldots, r_{n/2}\}$ and $B = \{b_1, b_2, \ldots, b_{n/2}\}$. If $R \cap B$ is nonempty, then we will reorder the edges so that $r_i = b_i$, for $1 \le i \le k-1$, where $k = |R \cap B| + 1$. As $R \neq B$, we must have $k - 1 \le n/2 - 2$.

The edges $r_k, \ldots, r_{n/2}$ and $b_k, \ldots, b_{n/2}$ must form red-blue alternating even cycles in G. If there exists t such cycles, then $\Phi(G) \ge 2^t$. Thus $t = 1$, and we denote this unique red-blue cycle by C.

Between any two edges of $r_1, \ldots, r_{k-1}$ there can exist at most two edges, and if there are two, then they are incident. If this were not the case, then we would contradict that $\Phi(G) = 2$. For the same reason, for any $1 \le i \le k-1$ there are at most two edges between r_i and any edge of C, and if two such edges exist they must be incident. Now consider $x, y \in V(C)$ and $xy \notin E(C)$. We call xy an *even chord* if $C \setminus \{x, y\}$ consists of two paths of even order, otherwise xy is an *odd chord*. Between any two red edges of C is at most one odd chord. When $|V(C)| \ge 6$ no even chord may exist, as otherwise a third 1-factor exists using the edge xy, the 1-factors of each even path of $C \setminus \{x, y\}$ and $r_1, \ldots, r_{k-1}$. Together these statements imply that only one edge may exist between any two red edges of C. In this way we have considered all edges with both ends in $V(C)$.

We thus obtain,

$$\begin{aligned}
|E(G)| &\le \overbrace{(k-1)}^{r_1,\ldots,r_{k-1}} + \overbrace{2\binom{k-1}{2}}^{r_i \text{ to } r_j} + \overbrace{(k-1)(n-2k+2)}^{r_1,\ldots,r_{k-1} \text{ to } C} + \overbrace{(n-2k+2)}^{E(C)} + \overbrace{\binom{n/2-k+1}{2}}^{\text{Chords of } C} \\
&= -\frac{1}{2}\left(k - \frac{n-1}{2}\right)^2 + \frac{n^2}{4} + \frac{9}{8} = g(k). \qquad (1)
\end{aligned}$$

Clearly, on the set $\{0, \ldots, \frac{n}{2} - 1\}$ the function $g(k)$ is maximized when $k = n/2 - 1$. Thus $|E(G)| \le g(n/2 - 1) = n^2/4 + 1$. This establishes the upper bound.

We now show the structure of the extremal graphs G. If the bound is to be achieved, then R and B intersect on $n/2 - 2$ edges and $V(C)$ induces $K_4 - e$. Also, the graph induced by the set $V(G) \setminus V(C)$ must be the graph H_{n-4}, with the edges of the unique 1-factor $r_1, \ldots, r_{n/2-2}$. Denote by v_i and w_i the endpoints of r_i in such a way that $v_1, \ldots, v_{n/2-2}$ and $w_1, \ldots, w_{n/2-2}$ are a clique and an independent set, respectively. We may also reorder the edges and assume that $d(v_i) < d(v_j)$ for $1 \le i < j \le n/2 - 2$.

If the bound is to be achieved, then there must exist four edges between a given r_i and $V(C)$ for $1 \le i \le n/2 - 2$. There are only two possible configurations of these edges; all others contradict $\Phi(G) = 2$. These two configurations are: (1) all four edges are incident with v_i, or (2) two edges are incident with w_i, two edges are incident with v_i and these four edges are incident only with (two) vertices of degree 3 in the $K_4 - e$. Note that we cannot have r_i with configuration type (1) and r_j with configuration type (2) for $i < j$. Otherwise, the vertices v_i, v_j, w_i, w_j and $V(C)$ yield a new 1-factor (with the edge $v_j w_i$). Consequently, a necessary condition for G to be an extremal graph is that the edges $r_1, \ldots, r_i$ have configuration type (2) and the edges $r_{i+1}, \ldots, r_{n/2-2}$ have configuration type (1) for some $0 \le i \le n/2 - 2$. Hence, there are at most $n/2 - 1$ such graphs. It is easy to see that the $n/2 - 1$ graphs G_n^i defined in the statement have this form and satisfy $\Phi(G_n^i) = 2$. This completes the proof. $\square$

One can easily generalize the proof of Theorem 2.2 to get the following.

Theorem 2.3 *If n is even and at least 4, then $f(n, 3) = \frac{n^2}{4} + 2$; otherwise $f(n, 3) = 0$. Furthermore, for each $n \ge 4$ there exists a unique extremal graph G_n; for $n = 4$ we have $G_n = K_4$, and for $n \ge 6$ we have $G_n = h(G_{n-2})$.*

Proof sketch. Since K_4 yields $f(4,2) = \frac{4^2}{4} + 2$, the lower bound follows from Lemma 2.1.

Now, let $\Phi(G) = 3$ and let R, B, and Y be the distinct 1-factors of G. As in the proof of Theorem 2.2, we consider R and B, and first consider the case when C has one even chord if $|V(C)| \geq 6$ or two odd chords if $|V(C)| = 4$. (If $|V(C)| \geq 6$ and C has more than one even chord, then $\Phi(G) > 3$.) In this case, Y is uniquely determined. Thus, we can again have a restriction on the number of edges similar to the above, i.e. an amount that is one more than found in the right side of (1). Otherwise, the chords of C must behave as in the proof of Theorem 2.2, and in at most one instance of the other pairwise comparisons between some r_i and an edge of C that we considered above we may have three edges present (but still only two edges between r_i, r_j for $1 \leq i < j \leq k-1$). Thus, we have a restriction on the number of edges that is one more than found in the right side of (1). In either case, we have

$$|E(G)| \leq -\frac{1}{2}\left(k - \frac{n-1}{2}\right)^2 + \frac{n^2}{4} + \frac{17}{8} = m(k).$$

Clearly, on the set $\{0, \ldots, \frac{n}{2} - 1\}$ the function $m(k)$ is maximized when $k = n/2 - 1$. Thus $|E(G)| \leq m(n/2-1) = n^2/4 + 2$. This establishes the upper bound.

We now show the structure of the extremal graph. In either case when $m(k)$ is maximized, $|V(C)| = 4$.

We first eliminate the latter case of yielding any extremal graphs. If equality in the above were to hold, then it must be that $V(C)$ induces $K_4 - e$ and that there exists some r_i for $1 \leq i \leq n/2 - 2$ such that three edges are present joining it and some edge of C. In addition, there must be two additional edges joining r_i and $V(C)$. A simple case analysis shows that regardless of the arrangement, we have $\Phi(G) > 3$.

In the former case if the bound is to be achieved, then $V(C)$ induces a K_4. Also, the edges $r_1, \ldots, r_{n/2-2}$ must induce H_{n-4}. Let us call the vertex belonging to r_i and the $(n/2-2)$-clique of this induced graph v_i and its partner vertex w_i for $1 \leq i \leq n/2 - 2$ — the set $\{w_1, \ldots, w_{n/2-2}\}$ is an independent set of size $n/2 - 2$. If the bound is to be achieved, then there must exist four edges joining a given r_i and $V(C)$. There is one possible configuration of these edges; all others contradict $\Phi(G) = 3$. The only configuration is that all four edges are incident with v_i. This yields the result. □

The approach taken in the proof of Theorem 2.2 cannot easily be generalized for determining $f(n,p)$ for $p \geq 4$, since the graph induced by 4 or more 1-factors may have structure richer than in the case when $p \leq 3$. In order to find $f(n,4)$, we have to use a different idea. Unfortunately, the new approach does not say too much about the structure of extremal graphs.

As a matter of fact, the quantitative parts of Theorem 2.2 and 2.3 can also be derived from the next lemma.

Lemma 2.4 *Let p be a positive integer. If $f(n,r) \leq C$ for every $1 \leq r \leq p$, then $f(n,p+1) \leq C+1$.*

Proof. Let G be an n-vertex graph with $\Phi(G) = p+1 \geq 2$ and $f(n,p+1)$ edges. To the contrary, we will assume that $f(n,p+1) > C+1$. We may find an edge e in G that belongs to at least one of the 1-factors but not to all of the 1-factors. Now consider $G - e$. The graph $G - e$ contains r

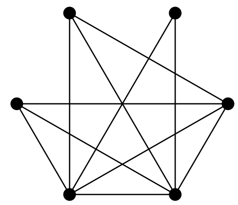

Figure 1: The unique 6-vertex graph with precisely four 1-factors and 11 edges.

1-factor(s) for some $1 \leq r \leq p$ and has precisely $f(n, p+1) - 1 > C \geq f(n, r)$ edges. This is a contradiction and establishes the lemma. □

Hetyei's Theorem and Lemma 2.4 immediately imply the following.

Corollary 2.5 *For every positive integer p, we have $f(n,p) \leq \frac{n^2}{4} + (p-1)$.*

Remark 2.6 It would be nice to prove Lemma 2.4 under a weaker condition, namely, assuming $f(n,p) \leq C$ only. Unfortunately, the function $f(n,p)$ is not monotonic in p. One can check[2] that $f(8,14) = 20 < 21 = f(8,12)$. Thus, in order to proceed in the proof of Lemma 2.4, we have to assume that $f(n,r) \leq C$ for $1 \leq r \leq p$.

In light of Hetyei's Theorem and Theorems 2.2 and 2.3 achieving the upper bound provided by Corollary 2.5, one might anticipate that it can always be achieved. We will show this not to be the case for $p \geq 4$.

Theorem 2.7 *If n is even and $n \geq 6$, then $f(n,4) = \frac{n^2}{4} + 2$; otherwise $f(n,4) = 0$.*

Proof. Let G_6 be the graph in Figure 1. Note that $\Phi(G_6) = 4$ and $|E(G_6)| = 11$. Consequently, we have $f(6,4) \geq \frac{6^2}{4} + 2$ and the lower bound follows from Lemma 2.1.

Let G be an n-vertex graph with $\Phi(G) = 4$. We will show that there exists an edge which is contained in either two or three 1-factors. (Hence, we may proceed in a similar fashion as in the proof of Lemma 2.4.) Suppose not. Consider the edges which belong to one 1-factor (they must exist) and denote the subgraph induced by them by H. Consider the union of any two 1-factors. The edges of these 1-factors that belong to H form the disjoint union of even cycles of girth at least four, i.e. they form a 2-factor. Consequently, this 2-factor must be a Hamiltonian cycle; otherwise we contradict $\Phi(G) = 4$. The same holds for the remaining two 1-factors. Hence H is a 4-regular graph which is the disjoint union of two Hamiltonian cycles. Thus, by a result of Thomason (Corollary 2.2 in [8]) H (and so G) contains at least 8 Hamiltonian cycles. On the other hand, the number of Hamiltonian cycles in G cannot exceed $\binom{\Phi(G)}{2} = 6$, since every Hamiltonian cycle is the union of two distinct 1-factors, a contradiction.

Now we may assume that there exists an edge e in G that belongs to exactly two or three 1-factors. Consider $G - e$. The graph $G - e$ contains precisely one or two 1-factor(s) and has exactly $|E(G)| - 1$ edges. Thus, by Hetyei's Theorem and Theorem 2.2 we obtain $|E(G)| - 1 \leq \frac{n^2}{4} + 1$, as required. □

Lemma 2.4 and Theorem 2.7 immediately improve the upper bound from Corollary 2.5.

Corollary 2.8 *If $p \geq 4$, then $f(n,p) \leq \frac{n^2}{4} + (p-2)$.*

[2]The authors used the program NAUTY provided by B. McKay, see http://cs.anu.edu.au/∼bdm/nauty/.

p	1	2	3	4	5	6
c_p	0	1	2	2	2	3
n_p	2	4	4	6	6	6

Table 1: $f(n,p) = \frac{n^2}{4} + c_p$ for n even and $n \geq n_p$.

With some additional effort, one can generalize previous results and show that $f(n,5) = \frac{n^2}{4} + 2$ and $f(n,6) = \frac{n^2}{4} + 3$. We omit these proofs since they are more technical and do not introduce any new ideas. We summarize our quantitative results in Table 1.

3 Open questions

In general, determining the value of $f(n,p)$ for an arbitrary p does not seem to be an easy problem. Notice that since the complete graph on $2t$ vertices contains $(2t-1)!!$ 1-factors, we have $f(n,(2t-1)!!) \geq \frac{n^2}{4} + (t^2 - t)$, which is tight for $t = 1$ and 2. It would be interesting to decide if this is also tight for any t.

Another intriguing question is to determine if the first inequality in Lemma 2.1 is always tight. If this would be the case, then $f(n,(2t-1)!!) = \frac{n^2}{4} + (t^2 - t)$.

There is also an interesting asymptotic aspect of function $f(n,p)$. Corollary 2.8 implies that $f(n,p) = \frac{n^2}{4} + O(p)$. Is the term $O(p)$ optimal? Is it true that $f(n,p) = \frac{n^2}{4} + o(p)$?

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