Minimum Degree and the Minimum Size of K_2^t-saturated Graphs

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Abstract

A graph G is said to be F-saturated if G does not contain a copy of F as a subgraph and $G+e$ contains a copy of F as a subgraph for any edge e contained in the complement of G. Erdős, Hajnal and Moon in [3] determined the *minimum* number of edges, $sat(n, F)$, such that a graph G on n vertices must have when F is a t-clique. Later, Ollmann [6] determined $sat(n, F)$ for $F = K_{2,2}$. Here we give an upper bound for $sat(n, F)$ when $F = K_2^t$ the complete t-partite graph with partite sets of size 2, and prove equality when G is of prescribed minimum degree.

Keywords: saturated graphs, minimum size, minimum degree

1 Introduction

We let $G = (V, E)$ be a graph on $|V| = n$ vertices and $|E| = m$ edges. We denote the complete graph on t vertices by K^t, and the complete multipartite graph with t partite sets each of size s by K_s^t. Let $F = (V', E')$ be a graph on $|V'| \leq n$ vertices. The graph G is said to be F-saturated if G contains no copy of F as a subgraph, but for any edge e in the complement of G, the graph $G + (e)$ contains a copy of F, where $G + (e)$ denotes the graph $(V, E \cup e)$. The celebrated theorem of Turán determines the *maximum* number of edges in a graph that is K^t-saturated. This number, denoted $ex(n, K^t)$, arises from the consideration of the so-called Turán graph. In 1964 Erdős, Hajnal and Moon [3] determined the *minimum* number of edges in a graph that is K^t-saturated. This number, denoted $sat(n, K^t)$, is $(t-2)(n-1) - \binom{t-2}{2}$ and arises from the split graph $K^{t-2} + \overline{K}^{n-t+2}$. Some years later Ollmann [6] determined the value $sat(n, K_{2,2})$. Tuza gave a shortened proof of this same result in [9]. Determining the exact value of this function for a given graph F has been quite difficult, and is known for relatively few graphs. Kászonyi and Tuza in [5] proved the best known general upper bound for $sat(n, F)$.

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We will say $u \sim v$ (respectively $u \not\sim v$) if $(uv) \in E(G)$ (respectively $(uv) \notin E(G)$). For any undefined terms we refer the reader to [1].

Theorem 1 (Kászonyi L. and Tuza, Z. [5]) *Let $\mathcal{F}$ be a family of non-empty graphs. Set*

$$u = min\{|U| : F \in \mathcal{F}, U \subset V(F), F - U \text{ is a star (or a star with isolated vertices)}\}$$

and

$$s = min\{|E(F - U)| : F \in \mathcal{F}, U \subset V(F), F - U \text{ is a star and } |U| = u\}.$$

Furthermore, let p be the minimal number of vertices in a graph $F \in \mathcal{F}$ for which the minimum s is attained. If $n \geq p$ then

$$sat(n, \mathcal{F}) \leq (u + \frac{s-1}{2})n - \frac{u(s+u)}{2}.$$

This result shows that $sat(n, \mathcal{F}) = O(n)$ where $\mathcal{F}$ is a family of graphs. Pikhurko [7] generalized this result to a family, $\mathcal{F}'$, of k-uniform hypergraphs by showing that $sat(n, \mathcal{F}') = O(n^{k-1})$. For a further summary of related results we refer the reader to [2].

Here we further refine the idea of $sat(n, F)$. To state the main result of this paper we define $sat(n, F, \delta)$ to be the *minimum* number of edges in a graph on n vertices and minimum degree δ that is F-saturated. We show the following two results.

Theorem 2 *For integers $t \geq 3$, $n \geq 4t - 4$,*

$$sat(n, K_2^t, 2t-3) = \lceil \frac{(4t-5)n - 4t^2 + 6t - 1}{2} \rceil.$$

This immediately implies the following.

Theorem 3 *For integers $t \geq 3$, $n \geq 4t - 4$,*

$$sat(n, K_2^t) \leq \lceil \frac{(4t-5)n - 4t^2 + 6t - 1}{2} \rceil.$$

It is worth noting that the bound provided by Theorem 3 is a slight improvement over that provided by Theorem 1. We also make the following conjecture.

Conjecture 1 *For integers $t \geq 3$, n sufficiently large, equality holds in Theorem 3.*

2 General Results

To prove Theorem 2 we will find the following results which are due to Tuza [9] to be useful.

Proposition 1 (Tuza [9]) *(a) If F is a k-vertex connected graph, other than the complete graph on k vertices, then every F-saturated graph G is $(k-1)$-vertex connected. (b) If F is a k-edge connected graph, then every F-saturated graph G is $(k-1)$-edge connected.*

Proposition 2 (Tuza [9]) *(a) Let F be a k-vertex connected graph, and let G be an F-saturated graph with a set X of $k-1$ vertices such that $G \setminus X$ is disconnected. Denote by $G_1, \ldots G_l$ the connected components of $G \setminus X$. If X induces a clique, then*
(1) $G \setminus G_i$ is F-saturated for $1 \leq i \leq l$;
(2) $G_i \cup X$ induces an F-saturated graph $1 \leq i \leq l$;
(b) Let F be a k-edge connected graph, and suppose that a graph G has a partition $V_1 \cup V_2 = V(G)$ such that there are just $k-1$ edges between V_1 and V_2. If G is F-saturated, then the subgraph induced by $V_i (i = 1, 2)$ is also F-saturated.

Proposition 3 *If G is a K_2^t-saturated graph $(t \geq 2)$ with cut-set X of order $2t-3$ and $G_1, G_2, \ldots, G_l$, are the components of $G \setminus X$, then all vertices belonging to X must belong to the K_2^t formed upon the addition of an edge $(v_i v_j)$ where $v_i \in G_i, v_j \in G_j (i \neq j)$. In other words there exist 3 vertices outside the cutset belonging to any such K_2^t formed. Additionally, 2 of these 3 vertices are in the same component of $G \setminus X$.*

Proof: Let G be a K_2^t-saturated graph. Let v_i, v_j be in separate components of $G \setminus X$. Consider $G + (v_i v_j)$. Clearly, there exists a vertex $z \neq v_i, v_j$ in some G_k belonging to the K_2^t formed upon the addition of edge $(v_i v_j)$ to G. Vertex z can not be in a component of $G \setminus X$ different from both v_i and v_j as then z would be non-adjacent to two vertices in the K_2^t-subgraph. Thus, without loss of gerenality z must be in say, G_i. Now suppose there exists another vertex w contained in the K_2^t in some $G_k, 1 \leq k \leq l$. Similarly, w must be in either G_i or G_j. If $w \in G_i$ then as v_j is not adjacent to both z and w, a K_2^t can not be formed, which is a contradiction. If $w \in G_j$ then as w is not adjacent to either v_i or z, again a K_2^t can not be formed, a contradiction. Hence, there are at most three vertices outside X (and thus exactly three vertices) in any such K_2^t and of these three vertices, two of them are in the same component of $G \setminus X$.□

Proposition 4 *If G is a K_2^t-saturated graph $(t \geq 2)$ with a cut-set X of order $2t-3$ then $X = \{x_1, x_2, \cdots x_{2t-3}\}$ induces a clique in G.*

Proof: Let G be a K_2^t-saturated graph as above and denote the components of $G \setminus X$ by $G_1, \cdots G_l$. Consider $G + (v_i v_j)$ where $v_i \in G_i, v_j \in G_j (i \neq j)$. By Proposition 3, the vertices of X are contained in the K_2^t formed upon inserting $(v_i v_j)$. Thus, on the vertices of X, a $K_2^{t-2} + x_k$ must be present in G. Now suppose there exists a pair of vertices x_i, x_j in X that are not adjacent in G. For any pair v_i, v_j as considered above, $G + (v_i v_j)$ contains a K_2^t where x_i and x_j must be in the same partite set. This implies that x_i, x_j are adjacent to all other vertices in the graph G. Thus $G \setminus \{x_i, x_j\}$ is K_2^{t-1}-saturated. Now consider $G + (x_i x_j)$. Upon the addition of edge $(x_i x_j)$ to G, a K_2^t is formed as a subgraph where x_i and x_j lie in different partite sets (as otherwise a K_2^t would have existed in G.) Thus, on $G \setminus \{x_i, x_j\}$ there exists a K_2^{t-1}, a contradiction.□

Proposition 5 *If G is a K_s^t-saturated graph with $t \geq 3$ ($t = 2$), then G has diameter at most 2 (respectively 3). Furthermore, if $t \geq 3$ then G contains $s(t-2)$ edge disjoint paths of length two between any two non-adjacent vertices.*

PROOF: Consider any pair of non-adjacent vertices x, y. Since every edge of K_s^t, $t \geq 3$ ($t = 2$) is contained in $s(t-2)$ 3-cycles (resp. a 4-cycle) and $G + (xy)$ contains the subgraph K_s^t, the distance from x to y in G can be no more than 2 (respectively 3.) □

Proposition 6 *If G is a K_2^t saturated graph with cut set X of order $2t-3$, then all vertices not adjacent to all of X belong to the same component of $G \setminus X$. Additionally, this component contains at least 3 vertices.*

PROOF: Consider vertices $v_i \in G_i, v_j \in G_j, i \neq j$ such that $v_i x_k \notin E(G)$ and $v_j x_l \notin E(G)$ for some $x_k, x_l \in X$ (note x_k may equal x_l). Now consider $G + (v_i v_j)$. By Proposition 3 there exists a vertex z in say G_i such that z is in the K_2^t formed upon the addition of edge $(v_i v_j)$ to G. But then v_j is not adjacent to both x_l and z, a contradiction. The same argument holds if z is in G_j. Thus v_i and v_j must be in the same component.

To see that this component has at least 3 vertices suppose that it did not. Then consider $G + (v_i x_k)$ and the K_2^t-subgraph formed. This copy of K_2^t must, by Proposition 2(2), lie entirely in X and this special component. But now we reach a contradiction, since X together with this component do not contain enough vertices.□

For convenience, from this point on we refer to the component described in Proposition 6 as G_1.

Proposition 7 *If G is a K_2^t-saturated graph with cut set X of order $2t-3$, then the components of $G \setminus X$ can be categorized as follows: (i) there is at most one component as described in Proposition 6, (ii) there is at most one component of order 1, and (iii) the remaining components are single edges.*

PROOF: (i) Follows immediately from Proposition 6. To show (ii), consider two components of order 1, say $G_i = \{a\}, G_j = \{b\}$. The graph G+(ab) must contain, by Proposition 3, a K_2^t on $X \cup \{a, b\}$. But this is impossible since $|X \cup \{a, b\}| = 2t - 1$. To show (iii) consider a component G_k where each vertex in G_k is adjacent to all of X and G_k contains at least 3 vertices. Note that in such a component there exists 3 vertices that induce at least two edges. This would imply the existence of a copy of K_2^t in G, which is a contradiction. Thus, these components have at most two vertices (and more than one) and therefore must be single edges. This proves (iii).□

Proposition 8 *If G is a K_2^t-saturated graph with cutset X of order $2t-3$, then any vertex v in G_1 is adjacent to at least $2t-4$ vertices of X.*

PROOF: Let $v \in G_1$ such that $vx_i \notin E(G)$ for some $x_i \in X$. Let w be in a different component, say G_j of $G \setminus X$. By Proposition 3, $G + (vw)$ contains a K_2^t which uses all of X. Hence, v must be adjacent to all other vertices of X. □

2.1 Proof of Main Result

We are now ready to prove the main result.

Proof of Theorem 2 : Let G be a K_2^t-saturated graph on $n \geq 4t-4$ vertices with $\delta(G) = 2t-3$.

We first note that in such a graph, $G + (v_1v_2)$ contains a copy of K_2^t where v_1 and v_2 are in different partite sets of K_2^t, as otherwise a copy of K_2^t would have already existed in G. If v_1 is in a partite set of K_2^t we will refer to the other vertex in that partite set as v_1's *mate*. For convenience we will refer to v_1 as being in the first partite set, v_2 the second partite set. Also, as K_2^t is a $(2t-2)$-connected graph, Proposition 1 implies that G is $(2t-3)$-connected, thus the minimum degree of any K_2^t-saturated graph is at least $2t-3$.

With reference to Proposition 7, we refer to a component of order 1 as a Type I component, a component of order 2 as a Type II component and a component of order 3 or more as a Type III component. Let y be a vertex of degree $2t-3$ and set $N(y) = X$. Note that X is a cut-set of size $2t-3$ and thus, by Proposition 4, the graph induced by X is complete. By Proposition 7 there is at most one component of Type III. Thus, there are two possibilities for the structure of G.

Case 1: Suppose G contains a component, G_1, of Type III

We begin by setting the number of vertices in G_1 equal to $g_1 \geq 3$, and describe the structure of G_1 and the minimum number of edges it must contain. First note that the number of Type II components is $k = \frac{n-2t+3-1-g_1}{2}$ (and thus n and g_1 have the same parity). Furthermore, by Proposition 2, $G_1 \cup X$ is a K_2^t-saturated graph. Denote by A the vertices of G_1 that are adjacent to all of X. Denote by X_1 the vertices of G_1 that are adjacent to $x_2, x_3, \cdots, x_{2t-3}$, but not x_1. Similarly, define X_i for $2 \leq i \leq 2t-3$. Note by Proposition 8, there are no other vertices of G_1. First note that if A is non-empty then A induces a 1-regular graph in G, since for any vertex $a \in A$, the graph $G + (ya)$ contains a K_2^t, and thus a must be adjacent to a vertex in A which is $y's$ mate. Further, there cannot exist two incident edges, say (a_1a_2) and (a_2a_3), in A as otherwise G would contain K_2^t as a subgraph. Namely a K_2^t would exist on $X \cup \{a_1, a_2, a_3\}$.

Furthermore, every vertex $v \in G_1 \setminus A$ is adjacent to exactly one vertex $a \in A$. To see this is true, first note that if $v \in G_1 \setminus A$ were adjacent to two vertices a_1, a_2 in A, then a K_2^t would be present in G, namely a K_2^t would exist on $X \cup \{v, a_1, a_2\}$. To see that v is adjacent to at least one vertex in A, note that $G + (vy)$ creates a K_2^t as a subgraph involving the $2t-1$ vertices $v, y, x_1, x_2, \cdots, x_{2t-3}$. The remaining vertex in the K_2^t subgraph which is not adjacent to y (as y has no other adjacencies in $G + (vy)$) must be $y's$ mate. Thus, this vertex must be adjacent to all others, which includes all of X, and thus this mate must be in A. This also shows that A cannot be empty. Together with the fact that A is 1-regular, this implies $|A| \geq 2$.

We now consider the maximum number of vertices $x \in V(G_1 \setminus A)$ such that $d_{G_1}(x) = 1$. Let $v, w \in G_1 \setminus A$ with $d_{G_1}(v) = d_{G_1}(w) = 1$. Then we consider the following two possibilities. Note that these conditions imply that $vw \notin E(G)$, as v's one edge in G_1 must be to A.

Subcase(i). Suppose $v, w \in X_i$ for some i, then the neighbors of v and w which are in A are adjacent.

Consider $G + (vw)$ and the K_2^t subgraph formed. The vertex x_i cannot be in the K_2^t formed as x_i is not adjacent to either v or w. This implies that v and w cannot share a single neighbor in A as then the joint neighborhood of v and w would contain only $2t-3$ vertices and any two non-adjacent vertices in G must have a joint neighborhood of at least $2t-2$ vertices. Thus suppose $v \sim a_1, w \sim a_2$ for some $a_1, a_2 \in A$. Additionally, $a_1 \sim a_2$ since the joint neighborhood is exactly $2t-2$ vertices and these two vertices lie in the symmetric difference of the joint neighborhood of v and w. In other words, a_1 is the mate of w and a_2 is the mate of v and thus the edge $(a_1 a_2)$ must exist.

Subcase (ii). Suppose $v \in X_i, w \in X_j,\ i \neq j$, then v and w share a common neighbor in A.

Without loss of generality suppose $v \in X_1, w \in X_2$. Further, suppose $v \sim a_1$ and $w \sim a_2$ for some $a_1, a_2 \in A, a_1 \neq a_2$. Now consider $G + (vw)$. Considering v, we see that the K_2^t formed must contain $v, w, a_1, x_2, x_3, \cdots, x_{2t-3}$. However, x_2 and a_1 are not adjacent to w, a contradiction. Therefore v, w must share the same neighbor in A.

For $t \geq 3$, (i) and (ii) together imply that the maximum number of vertices $x \in G_1$ such that $d_{G_1}(x) = 1$ is $2t-3$. Furthermore, this occurs when the $2t-3$ vertices are each in different X_i.

Once again we count the edges of G, and noting that $g_1 := |A| + |\cup_{i=1}^{2t-3} X_i|$. We explain the equation below. Beginning with line (1), recall that X is complete. Next, note that in this case each vertex in $G_2, G_3, \ldots G_l$ is adjacent to each vertex in X and that each of these Type II components contains one edge. Next line (2), each vertex in A is adjacent to all of X, and A induces a 1-factor. Next, each vertex in $\cup_{i=1}^{2t-3} X_i$ is adjacent to $2t-4$ vertices in X, and one vertex in A. Finally line (3), since there are at most $2t-3$ vertices, $\{u_1, u_2, \ldots u_{2t-3}\} \in \cup_{i=1}^{2t-3} X_i$ with $d_{G_1}(u_i) = 1$ the remainder must have degree at least two. Thus,

$$
\begin{aligned}
|E(G)| &\geq \binom{2t-3}{2} + (n-2t+3-g_1)(2t-3) + \frac{n-2t+3-1-g_1}{2} && (1)\\
&\quad + |A|(2t-3) + \frac{|A|}{2} + (|\cup_{i=1}^{2t-3} X_i|)(2t-4) + (|\cup_{i=1}^{2t-3} X_i|) && (2)\\
&\quad + \lceil \frac{(|\cup_{i=1}^{2t-3} X_i|) - min\{(2t-3), |\cup_{i=1}^{2t-3} X_i|\}}{2} \rceil && (3)\\
&= \lceil \frac{(4t-5)n - 4t^2 + 8t - 4 - min\{(2t-3), |\cup_{i=1}^{2t-3} X_i|\}}{2} \rceil && (4)
\end{aligned}
$$

and when $n \geq 4t-3$, the minimum is achieved when there exists at least $2t-3$ vertices in $\cup_{i=1}^{2t-3} X_i$. Thus,

$$|E(G)| \geq \lceil \frac{(4t-5)n - 4t^2 + 6t - 1}{2} \rceil. \quad (5)$$

Case 2: Suppose G contains no component of Type III.

If $n-2t+3$ is even (thus n is odd) then we reach a contradiction as $\frac{n-2t+2}{2}$ (the number, k, of Type II components) must be an integer. Thus $n-2t+3$ is odd and $k = \frac{n-2t+2}{2}$. We now count the number of edges G must contain. First, recall that X is complete. Next, note that in this case each vertex in $G \setminus X$ is adjacent to each vertex in X. Finally, note that each of the Type II components contains one edge. Thus,

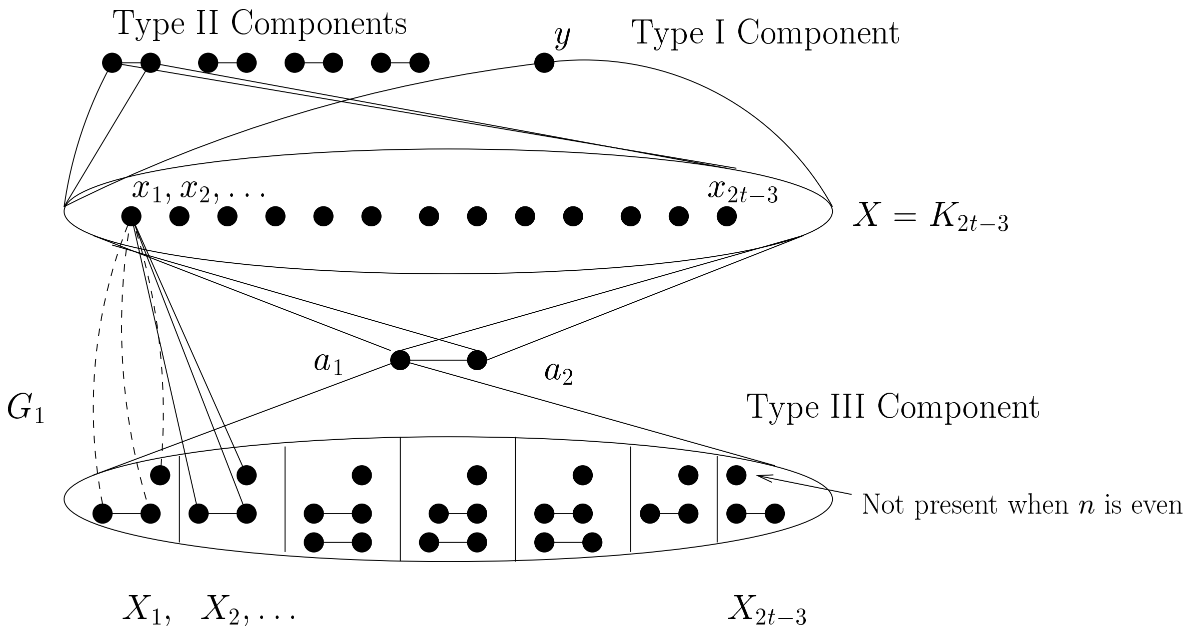

Figure 1: K_2^t-saturated graph

$$|E(G)| = \binom{2t-3}{2} + (n-2t+3)(2t-3) + \frac{n-2t+2}{2} \tag{6}$$

$$= \frac{(4t-5)n - 4t^2 + 8t - 4}{2}. \tag{7}$$

The number of edges obtained in the Case 1 is obviously less than in Case 2. We will now show that there exists a graph G that contains the number of edges as given by the lower bound in Case 1 and which is K_2^t-saturated.

It suffices to now describe the structure of G_1. The set A contains two adjacent vertices a_1, a_2, with a_1 adjacent to all of $\cup_{i=1}^{2t-3} X_i$. In the case that n is odd, each X_i contains a vertex u_i such that $d_{G_1}(u_i) = 1$. In the case that n is even, all but one of the X_i contain such a vertex. The remainder of the vertices in a given X_i induce a 1-factor. (That is we forbid edges $z_i z_j$ where $z_i \in X_i, z_j \in X_j, i \neq j$.) We have now completely described the structure of the graph G. Figure 1 helps to illustrate this.

We will now show that the minimal graph obtained in this case is indeed K_2^t-saturated, and thus the result will be established.

Claim 1 *The graph G contains no copy of K_2^t.*

First note that as the degree of y is $2t-3$, it cannot be contained in a copy of K_2^t. The same is true for any $u_i \in \cup_{i=1}^{2t-3} X_i$ such that $d_{G_1}(u_i) = 1$. If the copy of K_2^t contained all the vertices of X it would need to contain three vertices at distance two from y. These three vertices would need to be in the same component (as they must induce at least two edges), thus must be in G_1. If two vertices from A were used then there must exist some $v \in \cup_{i=1}^{2t-3} X_i$ that is adjacent to both of them as v is nonadjacent to some $x_i \in X$. However, v has only one edge to A. If one vertex of A were used, then the two remaining vertices, v, w can not come from the same X_i as $v, w \not\sim x_i$, and thus $v \in X_i, w \in X_j,\ i \neq j$. However, $v \not\sim x_i, w$ by construction. Thus all three vertices must come from $\cup_{i=1}^{2t-3} X_i$. Each would need to be in a different X_i, and thus must induce a triangle. However, this is forbidden from happening by our construction.

Thus, any copy of K_2^t would contain at most $2t-4$ vertices of X. Then at least 4 vertices of K_2^t must come from $G \setminus X$, and must be in the same component and thus lie in G_1. Furthermore, any four vertices of K_2^t contain a $K_{2,2}$ and a careful consideration of G_1 shows that no such $K_{2,2}$ exists. This proves the claim.□

Claim 2 *For any edge e in the complement of G, $G+e$ contains a copy of K_2^t.*

For convenience, let $a_1, a_2 \in A, z_{i,1}, z_{i,2} \in X_i, z_{j,1} \in X_j, v_j, w_j \in G_j, v_k \in G_k \quad (j,k \neq 1)$. We may assume that $d_{G_1}(z_{i,1}) = 2$ and will denote its neighbor in X_i by $z_{i,3}$. Also recall that for all $x \in \cup_{i=1}^{2t-3} X_i$ we have x adjacent to a_1.

To prove the claim we will show that for any edge e, the graph $G+e$ contains a copy of K_2^t and explicitly give each of the partite sets and their elements.

First we consider edges between components.

Case: Let $e = v_j v_k$, then K_2^t is contained in the subgraph induced by the following partite sets $\{\{w_j, v_k\}, \{v_j, x_1\}, \{x_2, x_3\}, \ldots \{x_{2t-4}, x_{2t-3}\}\}$.

Case: Let $e = v_k a_1$, then K_2^t is contained in the subgraph induced by the following partite sets $\{\{a_2, v_k\}, \{a_1, x_1\}, \{x_2, x_3\}, \ldots \{x_{2t-4}, x_{2t-3}\}\}$.

Case: Let $e = v_k a_2$, then K_2^t is contained in the subgraph induced by the following partite sets $\{\{a_1, v_k\}, \{a_2, x_1\}, \{x_2, x_3\}, \ldots \{x_{2t-4}, x_{2t-3}\}\}$.

Case: Let $e = v_k z_{i,1}$, then K_2^t is contained in the subgraph induced by the following partite sets $\{\{a_1, v_k\}, \{z_{i,1}, x_i\}, \{x_1, x_2\}, \ldots \{x_{2t-4}, x_{2t-3}\}\}$.

Next we consider edges from the cut-set to G_1.

Case: Let $e = x_i z_{i,2}$, then K_2^t is contained in the subgraph induced by the following partite sets

$$\{\{z_{i,2}, a_2\}, \{x_i, a_1\}, \overbrace{\{x_1, x_2\}, \ldots \{x_{2t-4}, x_{2t-3}\}}^{omits\ \ x_i}\}.$$

This leaves us to consider edges within G_1.

Case: Let $e = a_2 z_{i,2}$, then K_2^t is contained in the subgraph induced by the following partite sets

$$\{\{z_{i,2}, x_i\}, \{a_1, a_2\}, \overbrace{\{x_1, x_2\}, \ldots \{x_{2t-4}, x_{2t-3}\}}^{omits\ \ x_i}\}.$$

Case: Let $e = z_{i,1}z_{i,2}$, then K_2^t is contained in the subgraph induced by the following partite sets $\{\{z_{i,1}, a_1\}, \{z_{i,2}, z_{i,3}\}, \overbrace{\{x_1, x_2\}, \ldots \{x_{2t-4}, x_{2t-3}\}}^{omits\ \ x_i}\}$.

Case: Let $e = z_{i,1}, z_{j,1}$, then K_2^t is contained in the subgraph induced by the following partite sets $\{\{z_{i,1}, x_i\}, \{z_{j,1}, x_j\}, \{a_1, x_1\}, \overbrace{\{x_2, x_3\}, \ldots \{x_{2t-4}, x_{2t-3}\}}^{omits\ \ x_i,\ x_j,\ x_1}\}$.

This completes the proof of Claim 2, and the proof of Theorem 2.□

We now give further evidence to support Conjecture 1. To do this we begin by generalizing a Theorem used by Duffus and Hanson in [4].

Theorem 4 *For integers $t \geq 3, s \geq 1, \delta \geq s(t-1)-1,\ n \geq st$,*

$$sat(n, K_s^t, \delta) \geq \frac{\delta + s(t-2)}{2}(n - \delta - 1) + \delta + s^2\binom{t-2}{2} + s(s-1)(t-2). \tag{8}$$

PROOF: Let y be a vertex of minimum degree δ and X the set of δ vertices adjacent to y. Let Z denote the remaining $n-\delta-1$ vertices, which are at distance two (by Proposition 5) from y. First, X contains a copy of $K_s^{t-2} + \overline{K}_{s-1}$ since $G + (yv)$ contains a K_s^t, $v \in Z$, for any $v \not\sim y$. Next, each $v \in Z$ must be adjacent to all of the vertices of a K_s^{t-2} in X since $G + (yv)$ creates a copy of K_s^t. Therefore, by summing the degrees of the vertices in each set we obtain,

$$\begin{aligned}\Sigma_{x \in G} d(x) \quad \geq \quad & \delta + \{\delta + s(t-2)(n-\delta-1) + s(t-2)[s(t-3) + (s-1)] + (s-1)[s(t-2)]\} \\ & + \{(n-\delta-1)\delta\}.\end{aligned}$$

The lower bound thus follows.□

We now use Theorem 4 in support of Conjecture 1. Evaluating Equation 8 for $s = 2$ and $\delta \geq 2t$ we find that the coefficient in n is at least $\frac{4t-4}{2}$ which is greater than the coefficient in n given by Theorem 2, which is $\frac{4t-5}{2}$. Thus for n sufficiently large the number of edges in an K_2^t-saturated graph with minimum degree $\delta \geq 2t$ is strictly greater than the number of edges in an K_2^t-saturated graph with minimum degree $2t-3$.

This leads to another conjecture (which generalizes one given by Bollobás in [2]), the proof of which would settle Conjecture 1.

Conjecture 2 *Given a fixed graph F, for n sufficiently large the function $sat(n, F, \delta)$ is monotonically increasing in δ.*

We note that the word "monotonically" can not be replaced by "strictly." One can see this by examining the extremal graphs for $K_{2,2}$ provided by Ollmann [6].

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