Journal of Combinatorics
, , ,

Saturation Numbers for Families of Ramsey-minimal Graphs

Guantao Chen, Michael Ferrara, Ronald J. Gould, Colton Magnant and John Schmitt*

Given a family of graphs $\mathcal{F}$, a graph G is $\mathcal{F}$*-saturated* if no element of $\mathcal{F}$ is a subgraph of G, but for any edge e in $\overline{G}$, some element of $\mathcal{F}$ is a subgraph of $G+e$. Let $sat(n,\mathcal{F})$ denote the minimum number of edges in an $\mathcal{F}$-saturated graph of order n.

For graphs $G, H_1, \dots, H_k$, we write that $G \to (H_1, \dots, H_k)$ if every k-coloring of $E(G)$ contains a monochromatic copy of H_i in color i for some i. A graph G is $(H_1, \dots, H_k)$-*Ramsey-minimal* if $G \to (H_1, \dots, H_k)$ but for any $e \in G$, $(G-e) \not\to (H_1, \dots, H_k)$. Let $\mathcal{R}_{\min}(H_1, \dots, H_k)$ denote the family of $(H_1, \dots, H_k)$-Ramsey-minimal graphs.

In 1987, Hanson and Toft conjectured that

$$sat(n, \mathcal{R}_{\min}(K_{k_1}, \dots, K_{k_t})) =$$

$$\begin{cases} \binom{n}{2} & n < r \\ \binom{r-2}{2} + (r-2)(n-r+2) & n \geq r, \end{cases}$$

where $r = r(k_1, k_2, \dots, k_t)$ is the classical Ramsey number for cliques.

In this paper, we settle the first non-trivial case of Hanson and Toft's conjecture for sufficiently large n by showing that $sat(n, \mathcal{R}_{\min}(K_3, K_3)) = 4n-10$ for $n \geq 56$. We also undertake a brief investigation of $sat(n, \mathcal{R}_{\min}(K_t, T_m))$ where T_m is a tree of order m.

AMS 2000 subject classifications: Primary 05C35, 05C55.

Keywords and phrases: Ramsey-minimal, saturation number.

*This material is based on research done while supported by an ROA Supplement to National Science Foundation (NSF) grant DMS-0758057.

1. Introduction

In this paper we consider only graphs without loops or multiple edges. We let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively and we will let $e(G) = |E(G)|$. For any vertex v in G, let $N(v)$ and $N[v] = N(v) \cup \{v\}$ denote the neighborhood and closed neighborhoods of v, respectively. We denote the complement of G by $\overline{G}$. Given any two graphs G and H, their *join*, denoted $G \vee H$, is the graph with $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{gh \mid g \in V(G),\ h \in V(H)\}$. Finally, for a set of vertices X in G, let $\langle X \rangle_G$ denote the subgraph of G induced by X.

Given a family of graphs $\mathcal{F}$, a graph G is *$\mathcal{F}$-saturated* if no element of $\mathcal{F}$ is a subgraph of G, but for any edge e in $\overline{G}$, some element of $\mathcal{F}$ is a subgraph of $G + e$. If $\mathcal{F} = \{H\}$, then we say that G is *H-saturated*. The classical extremal function $ex(n, H)$ is precisely the maximum number of edges in an H-saturated graph of order n. Erdős, Hajnal and Moon [5] studied $sat(n, H)$, the *minimum* number of edges in an H-saturated graph, and determined $sat(n, K_t)$.

Theorem 1 *Let n and t be positive integers such that $n \geq t$. Then*

$$sat(n, K_t) = \binom{t-2}{2} + (t-2)(n-t+2).$$

Furthermore, $K_{t-2} \vee \overline{K}_{n-t+2}$ is the unique K_t-saturated graph of order n with minimum size.

For graphs $G, H_1, \ldots, H_k$, we write that $G \to (H_1, \ldots, H_k)$ if every k-coloring of $E(G)$ contains a monochromatic copy of H_i in color i for some i. A graph G is $(H_1, \ldots, H_k)$*-Ramsey-minimal* if $G \to (H_1, \ldots, H_k)$ but for any $e \in G$, $(G - e) \not\to (H_1, \ldots, H_k)$. Let $\mathcal{R}_{\min}(H_1, \ldots, H_k)$ denote the family of $(H_1, \ldots, H_k)$-Ramsey-minimal graphs.

In this paper, we consider $sat(n, \mathcal{R}_{\min}(H_1, \ldots, H_k))$ for certain choices of the H_i. It is straightforward to show that any graph G such that $G \to (H_1, \ldots, H_k)$ must contain a Ramsey-minimal subgraph. Therefore, determining the saturation number for $\mathcal{R}_{\min}(H_1, \ldots, H_k)$ is equivalent to determining the minimum number of edges in a graph G of order n with the property that $G \not\to (H_1, \ldots, H_k)$ but $G + e \to (H_1, \ldots, H_k)$ for any edge e in the complement of G. In 1987, Hanson and Toft [8] discussed this notion and made the following conjecture.

Conjecture 1 *Let $r = r(k_1, k_2, \ldots, k_t)$ be the standard Ramsey number for complete graphs. Then*

$$sat(n, \mathcal{R}_{\min}(K_{k_1}, \ldots, K_{k_t})) = \begin{cases} \binom{n}{2} & n < r \\ \binom{r-2}{2} + (r-2)(n-r+2) & n \geq r. \end{cases}$$

The statement of this conjecture can also be found in [9].

For $n \geq r$, the fact that $sat(n, \mathcal{R}_{\min}(K_{k_1}, \ldots, K_{k_t})) \leq \binom{r-2}{2} + (r-2)(n-r+2)$ arises from consideration of the graph $G = K_{r-2} \vee \overline{K}_{n-r+2}$ which, by Theorem 1 is the unique K_r-saturated graph of minimum size. Consequently, for any $e \in \overline{G}$, $G + e$ contains K_r and thus $G + e \to (K_{k_1}, \ldots, K_{k_t})$.

To see that $G \not\to (K_{k_1}, \ldots, K_{k_t})$, consider the coloring obtained by cloning a vertex in any edge-coloring of K_{r-1} containing no K_{k_i} in color i. For an example when $k_1 = k_2 = 3$ see Figure 1.

The main result of this paper is that $sat(n, \mathcal{R}_{\min}(K_3, K_3)) = 4n - 10$ for $n \geq 56$, settling the smallest non-trivial case of Conjecture 1 for sufficiently large n. We also undertake an investigation of the parameter $sat(n, \mathcal{R}_{\min}(K_t, T_m))$ where T_m is a tree of order m.

2. Main Result

We now proceed by giving our main result.

Theorem 2 *For $n \geq 56$,*

$$sat(n, \mathcal{R}_{\min}(K_3, K_3)) = 4n - 10.$$

Proof: The fact that $sat(n, \mathcal{R}_{\min}(K_3, K_3)) \leq 4n - 10$ follows from the coloring of $K_4 \vee \overline{K}_{n-4}$ described above and pictured in Figure 1.

We therefore aim to prove that $sat(n, \mathcal{R}_{\min}(K_3, K_3)) \geq 4n - 10$.

Let G be a $\mathcal{R}_{\min}(K_3, K_3)$-saturated graph of order n and suppose that $e(G) < 4n - 10$. Fix a coloring χ of $E(G)$ that contains neither a red nor blue K_3. Throughout the proof, we will attempt to add red or blue edges to G. We demonstrate that by modifying χ (much as Galluccio et al. [7] did in their paper), we may add an edge without creating a monochromatic K_3, violating our assumption that G is $\mathcal{R}_{\min}(K_3, K_3)$-saturated.

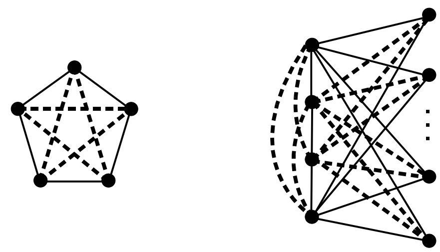

Figure 1: A coloring of K_5 with no monochromatic triangles that gives rise to a $\mathcal{R}_{\min}(K_3, K_3)$-saturated graph.

It is convenient to visualize G with respect to the coloring χ, and as such we will frequently refer to the "blue graph" and "red graph", as well as the "blue" or "red" degree of a vertex. More formally, let G_{red} and G_{blue} denote the subgraphs of G consisting of the edges colored red and blue by χ, respectively. Similarly, for a vertex v in G, $d_r(v)$ will denote the number of red edges incident to v and $N_{red}(v)$ denotes the "red neighborhood" of v, that is, the set of vertices u such that uv is red. Similarly, let $N_{red}[v]$ denote $N_{red}(v) \cup v$. We define $d_b(v), N_{blue}(v)$ and $N_{blue}[v]$ in an analogous manner with respect to the blue graph G_{blue}. We will refer to a vertex x in $N_{red}(v)$ as a *red neighbor* of v and will also say that the vertices v and x are *red-adjacent*.

We now explore the structure of $G = G_{red} \cup G_{blue}$, and work toward a contradiction.

Claim 1 *The graphs G_{red} and G_{blue} are connected.*

Proof: Without loss of generality, suppose that G_{red} is disconnected, and let A be a component of G_{red} having minimum order. Note that for every pair of vertices $a \in A$ and $b \in B = G_{red} - A$, the edge ab must be in G_{blue}, since otherwise we could add a missing edge in red without creating a red K_3. The fact that all of these edges are blue also yields that there are no blue edges within $\langle A \rangle_G$ and $\langle B \rangle_G$ and hence that both of these (red) subgraphs are K_3-saturated. Utilizing Theorem 1, it follows that

$$e(G) \geq |A||B| + (|A| + |B| - 2).$$

Since $e(G) < 4n - 10$ and $n \geq 56$, we conclude that $|A| \leq 3$.

Since every edge within B is red and $|B| \geq n - 3$, there is some edge uv not in B. We will add this edge uv, colored blue, to G and then modify

χ to remove any blue triangles. Suppose $|A| = 3$, and note that since $\langle A \rangle_G$ is K_3-saturated, $A \cong P_3$. If we label the vertices of this P_3, in order, with a_1, a_2 and a_3, then after adding uv in blue, we recolor ua_1, va_2 and ua_3 red. This does not create a monochromatic triangle, and hence contradicts the assumption that G is $\mathcal{R}_{\min}(K_3, K_3)$-saturated. The cases where $|A| < 3$ are handled in a similar manner. □

Since G is $\mathcal{R}_{\min}(K_3, K_3)$-saturated, the addition of an edge colored red or blue must create a triangle of that color. The following fact reflects this observation.

Fact 1 *If u and v are nonadjacent vertices in G, then $N_{red}(u) \cap N_{red}(v)$ and $N_{blue}(u) \cap N_{blue}(v)$ are both non-empty.*

Also note that if $\alpha(G) \geq n-4$ then G is a subgraph of $K_4 \vee \overline{K}_{n-4}$. Thus any edge in $E(K_4 \vee \overline{K}_{n-4})$ that is not contained in $E(G)$ could be added to G without destroying the coloring depicted in Figure 1, contradicting the assumption that $e(G) < 4n - 10$. This implies the $\alpha(G) < n - 4$.

For the remainder of the proof, we let v denote a vertex of minimum degree in G and we let H denote $G - N[v]$. Fact 1 and the assumption that $e(G) < 4n - 10$ imply that $2 \leq d(v) \leq 7$.

Galluccio, Simonovits, and Simnoyi [7] investigated $\mathcal{R}_{\min}(K_3, K_3)$-saturated graphs, though not just those of minimum size. They gave various constructions of such graphs and various structural results. Useful in establishing our next claim is the following theorem.

Theorem 3 *[7] If G is a $\mathcal{R}_{\min}(K_3, K_3)$-saturated graph G, then $\delta(G) \geq 4$.*

Claim 2 *$d(v) \geq 4$.*

Proof: Follows immediately from Theorem 3. □

Let x be a vertex in H and let v' be a vertex in $N(v) \cap N(x)$ such that the edges $v'v$ and $v'x$ are different colors. In this situation, the edge $v'x$ does not prevent us from inserting the edge xv in either color, so we call such an edge a *wasted edge to* $N(v)$. Additionally, say that an edge from x to $N(v)$ is *useful* if it is not wasted.

Claim 3 *$d(v) \leq 5$.*

Proof: To begin, suppose that $\delta(G) = d(v) = 7$. Every vertex in H must have a common red neighbor and a common blue neighbor with v. This implies that

$$e(G) \geq 7 + \frac{1}{2}\left(\sum_H d(x) + e(N(v), H)\right)$$

$$\geq 7 + \frac{1}{2}\left(7(n-8) + 2(n-8)\right),$$

which is at least $4n - 10$ for $n \geq 38$.

Therefore assume that $\delta(G) = d(v) = 6$. If the subgraph induced by $N(v)$ does not contain a red edge, then we can recolor all six edges incident with v red without a red or a blue K_3, which contradicts Claim 1. We could similarly recolor and obtain a contradiction if the subgraph induced by $N(v)$ contained no blue edges. Consequently, there must be at least one blue edge and at least one red edge in the subgraph induced by $N(v)$ and hence at least eight edges in the subgraph induced by $N[v]$.

Consider a vertex x in H such that $|N(v) \cap N(x)| = 2$. By Fact 1, x and v must have a common red neighbor r and a common blue neighbor b. Suppose that the edge br is not red (meaning that it may not be in G at all).

Recoloring xb red would leave x and v with no blue common neighbor, so there must be a vertex v_r in G that is red-adjacent to both x and b. Note that since rb is not red, $r \neq v_r$, so $v_r \in H$. We also cannot recolor vb red, so there must be a vertex $r_2 \neq r$ in $N_{red}(v)$ such that r_2b is red. Since xr_2 is not in G, we must not be able to add it to G_{blue}. Hence there must be some vertex v_b in H that is blue-adjacent to both x and r_2. Note that $v_b \neq b$, as r_2b is red. We can therefore conclude that for every vertex x having exactly two common neighbors with v, there are two vertices in $N(x) - N(v)$, each having a wasted edge.

Every vertex in G has degree at least six, and every vertex in H has at least two edges to $N(v)$. Summing the degrees in G, we get that

$$2e(G) \geq 16 + 2(n-7) + 6(n-7) + \sum_H (d_{N(v)}(x) - 2) + \sum_H (d(x) - 6)$$

$$= 8n - 40 + \sum_H (d_{N(v)}(x) - 2) + \sum_H (d(x) - 6).$$

The vertices v_r and v_b described above each have at least two useful and at least one wasted edge to $N(v)$, hence they each contribute at least one to $\sum_H (d_{N(v)}(x) - 2)$. Furthermore, each of these vertices may be used to prevent recoloring with respect to a number of choices of x. Since v_r and v_b must be adjacent to x, they may prevent recoloring with respect to at most three choices of x before beginning to contribute to $\sum_H(d(x)-6)$. Consequently, if we let n_2 denote the number of vertices in H with exactly two neighbors in $N(v)$, we get that

$$\sum_H(d_{N(v)}(x)-2)+\sum_H(d(x)-6)\geq\frac{2n_2}{3}$$

or that

$$e(G)\geq 4n-20+\frac{n_2}{3}.$$

Since $e(G)<4n-10$, it follows that $n_2\leq 29$.

There are $n-7-n_2$ vertices with three or more neighbors in $N(v)$, implying that

$$2e(G)\geq 6(n-7)+2(n-7)+(n-n_2-7)+16,$$

which is a contradiction, given that $n_2\leq 29$ and $e(G)<4n-10$. $\square$

Claims 2 and 3 together imply that $\delta(G)$ is either four or five. Next, we eliminate the former possibility.

Claim 4 $\delta(G)=5$.

Proof: Assume that $d(v)=4$ and begin by assuming that $N_{blue}(v)=\{b\}$ and $N_{red}(v)=\{r_1,r_2,r_3\}$. Note that in order to avoid a monochromatic K_3, some pair of red neighbors of v must be nonadjacent, say r_1 and r_2. Also note that, by Fact 1, b must be blue-adjacent to every vertex in H. We may therefore assume that each br_i is red since the only red neighbors of b lie in $N_{red}(v)$, which contains no red edges. Now, since r_1 and r_2 are nonadjacent, we may recolor the edges vr_1 and vr_2 blue. This forces, by Fact 1, r_3 to be red-adjacent to every vertex x in H. Consequently, H must be an independent set. Thus $H\cup\{v\}$ is an independent set of order $n-4$, a contradiction.

Hence, we may assume that $N_{red}(v)=\{r_1,r_2\}$, $N_{blue}(v)=\{b_1,b_2\}$ and that we cannot recolor the edges incident to v such that G contains no monochromatic K_3 and v is incident to at most one edge of some color. Along these lines, to prevent us from recoloring vb_1 red, b_1 must have a red edge to one of the vertices in $N_{red}(v)$, say r_1. Then, to prohibit us from coloring vr_1 blue, r_1b_2 must be in G_{blue}. Similarly, we conclude that b_2r_2 must be in

G_{red} and b_1r_2 must be in G_{blue}. By Fact 1, every vertex of H must have a red neighbor in $N_{red}(v)$ and a blue neighbor in $N_{blue}(v)$. Hence, the addition of the blue edge r_1r_2 and the red edge b_1b_2 cannot create a monochromatic triangle, so these edges must be present in these colors. Together, this means that the subgraph induced by $N[v]$ is a complete graph composed of disjoint monochromatic 5-cycles, $vb_1r_2r_1b_2v$ in blue and $vr_1b_1b_2r_2v$ in red.

Since $N(v)$ is complete and $e(G) < 4n - 10$, there must be a vertex x in H with three or fewer neighbors in $N(v)$. Suppose that $d_{N(v)}(x) = 2$, specifically that xr_1 is red and xb_1 is blue. Since r_1b_1 is red, we cannot recolor xb_1 red, so we will attempt to recolor xr_1 blue instead. To prevent this, there must be a vertex x_b, necessarily in H, such that r_1x_b and x_bx are both in G_{blue}. Since x_b has a red neighbor in $N_{red}(v)$, x_br_2 must be red. Let $x_1, x_2, \ldots$ enumerate all possible choices for x_b in G, and sequentially recolor x_ix red if possible. As we cannot recolor all such edges without forcing a contradiction, we may assume that x_bx cannot be recolored red. This implies the existence of a vertex y, also in H, such that yx and yx_b are both red. Since xr_1 and x_br_2 are both red, this prohibits y from having a red neighbor in $N_{red}(v)$, a contradiction.

Suppose then that $d_{N(v)}(x) = 3$, specifically that xb_1 is blue and that xb_2 and xr_1 are both red (so that xb_2 is wasted). We cannot recolor xr_1 blue, and since r_1b_1 is red, there must be a vertex x_b in H such that x_bx and x_br_1 are both in G_{blue}. Therefore, x_br_2 must be red and, if we enumerate all possible options for x_b as above and sequentially recolor, there must be some choice of x_b such that x_bx cannot be recolored red. Thus there is a y in H such that y is red-adjacent to both x_b and x and once again y cannot have a red neighbor in $N_{red}(v)$.

Finally, we may assume that x has no wasted edges to $N(v)$, so that (without loss of generality) xr_1 and xr_2 are red, while xb_1 is blue. If we could recolor xr_1 blue, then x would have a wasted edge, reducing to the previous case. Hence there is some vertex x_b such that x_bx and x_br_1 are both blue, and since b_1r_1 is red, x_b must be in H. Now since xb_1 and b_2r_1 are both blue, x_b cannot be blue-adjacent to b_1 or b_2, the final contradiction necessary to complete the claim. $\square$

Therefore, $d(v) = \delta(G) = 5$.

Claim 5 *The vertex v is incident to at least two edges of each color.*

PROOF: Suppose otherwise, and let $N_{red}(v) = \{r_1, \ldots, r_4\}$ and $N_{blue}(v) = \{b\}$. By Fact 1, b is blue-adjacent to every vertex in H. Moreover, as in the previous claim, we may assume that each of the edges br_i is in G_{red}.

Let B denote the subgraph induced by $N_{red}(v)$, necessarily a subgraph of G_{blue}, which must be triangle-free and hence is bipartite.

Case 1: Suppose $\alpha(B) \geq 3$.

Assume, without loss of generality, that r_2, r_3 and r_4 are independent in B. Since we have assumed that b is red-adjacent to each r_i, we could recolor each of vr_2, vr_3 and vr_4 blue without creating a blue triangle. This implies that r_1 is red-adjacent to every vertex in H and furthermore, since $H \subseteq N_{blue}(b)$, that H is an independent set. Consequently we could add, in red, any missing edge from r_2, r_3 or r_4 to H without creating a red triangle, so all of these edges must be present in G. We conclude that G has at least $5(n-6)$ edges, a contradiction for $n \geq 20$.

Case 2: Suppose $B \subseteq K_{2,2}$.

Specifically, let $\{r_1, r_2\}$ and $\{r_3, r_4\}$ be independent sets in B. We could recolor vr_1 and vr_2 blue, so every vertex x in H must have a red edge to one of r_3 or r_4. Symmetrically, each x in H must have a red edge to one of r_1 or r_2. Choose some x in H and assume that x is red-adjacent to r_1 and r_3 and furthermore that xr_2 is not in G. The only way we are prohibited from adding the edge xr_2 in blue is if x and r_2 have a common blue neighbor. Since $H \subseteq N_{blue}(b)$, there are no blue edges in H, so this common blue neighbor must be in $N(v)$. Specifically r_4 must be the common blue neighbor of x and r_2. Consequently, every vertex x in H must have four neighbors in $N(v)$. This implies that

$$
\begin{aligned}
2e(G) &\geq \sum_{x \in H} d(x) + e(N(v), H) \\
&\geq 5(n-6) + 4(n-6) = 9n - 54,
\end{aligned}
$$

contradicting our assumption that $e(G) < 4n - 10$ when $n \geq 34$. □

For the remainder of the proof of Theorem 2, we will assume that $N_{blue}(v) = \{b_1, b_2\}$ and $N_{red}(v) = \{r_1, r_2, r_3\}$.

Claim 6 *Every vertex x in H is adjacent to at least three vertices in $N(v)$.*

Proof: By Fact 1, each vertex in H is red-adjacent and blue-adjacent to at least one vertex in $N(v)$. Suppose that x in H is red-adjacent to r_1, blue-adjacent to b_1 and has no other neighbor in common with v. Suppose first that b_1r_1 is not in G_{blue}. Note that by Fact 1 we cannot recolor xr_1 blue, so there must be some vertex x_b in H that is blue-adjacent to both x and r_1. Then x_b cannot be blue-adjacent to b_1, implying that x_bb_2 must be in G_{blue}. However, we cannot recolor vr_1 blue since x has only one red neighbor in

$N(v)$, so r_1 must be blue adjacent to either b_1 or b_2. Since we assumed r_1b_1 was not in G_{blue}, this implies that r_1 is blue-adjacent to b_2, creating a blue triangle with x_b.

Therefore, we may assume that r_1b_1 is blue. We cannot recolor xb_1 red, so there must be some vertex y in H that is red-adjacent to both x and b_1. This implies that yb_2 is blue. Once again, let $y_1, y_2, \ldots$ enumerate all possible choices of y, and sequentially recolor the edges xy_i blue if possible. As we may not recolor all of these edges, we assume that we have chosen y so that xy cannot be recolored blue. This implies the existence of a vertex y_b in H such that y_by and xy_b are both blue. However, as in the above claims, y_b cannot be blue-adjacent to b_1 or b_2, a contradiction. □

Claim 7 *If a vertex x in H has a wasted edge to $N(v)$, then at least one of the following holds*

1. $d_{N(v)}(x) \geq 4$,
2. $d(x) \geq 6$ *or*
3. $d(x) = 5$ *and* $d_{N(v)}(x) = 3$. *Also, the two vertices in $N_H(x)$ are adjacent and at least one $y \in N_H(x)$ has degree at least six.*

Proof: Let x be a vertex in H with a wasted edge to $N(v)$ and suppose that conditions (1) and (2) do not hold.

Case 1: Suppose x has a wasted blue edge.

Assume, without loss of generality, that xr_1 is red and both xb_1 and xr_2 are blue. Furthermore, we first assume that r_1b_1 is not in G_{blue}. We cannot recolor xr_1 blue, so there must be a vertex x_b that is blue-adjacent to both x and r_1. Suppose that x_b is in H, and note that if x_bb_1 was blue we would have a blue K_3, so x_bb_2 must be in G_{blue}. Since x has exactly three neighbors in $N(v)$, the edge vr_1 cannot be recolored blue, so r_1 must be blue-adjacent to some vertex in $N_{blue}(v)$. Since r_1 and b_2 are both blue-adjacent to x_b, we conclude that r_1 must be blue-adjacent to b_1, contradicting our assumption that r_1b_1 is not in G_{blue}. Hence, we may suppose that $x_b = r_2$ meaning that r_1r_2 is blue.

We cannot recolor vr_1 blue, as r_1 is the only red neighbor of x. Thus r_1 must have a blue neighbor in $N_{blue}(v)$, so by assumption r_1b_2 must be in G_{blue}. As b_1b_2 is not in G_{blue}, the addition of xb_2 to G in blue forces there to exist a vertex x_b that is blue-adjacent to both b_2 and x. Note that $x_b \neq r_2$ since the blue edge r_2b_2 would create a blue triangle $r_2b_2r_1$, so $x_b \in H$. Similarly, as xr_3 is not in G the addition of xr_3 in red forces the existence of a vertex y that is red-adjacent to both r_3 and x. Again note that $y \neq r_1$ since there can be no red edge within $N_{red}(v)$, so $y \in H$.

By assumption, x has no more neighbors. Therefore, to prohibit the addition of xb_2 to G_{red}, x and b_2 must have a common red neighbor. Since r_1b_2 was assumed to be blue, this means that yb_2 is in G_{red} and hence that yb_1 is in G_{blue}.

Now we wish to show that $d(y) \geq 6$ and $d_{N(v)}(y) \geq 3$. The latter holds as we have shown that yr_3, yb_1 and yb_2 are in G. If y is adjacent to both r_1 and r_2, then $d(y) \geq 6$, so suppose first that y is adjacent to neither r_1 nor r_2. To prevent adding either yr_1 or yr_2 to G_{blue}, there must be vertices y_1 and y_2 such that y_1 is blue-adjacent to both r_1 and y, and y_2 is blue-adjacent to both r_2 and y. Note that since r_1r_2 is blue, $y_1 \neq y_2$ and also that $y_2 \neq b_1$ as x is blue-adjacent to both r_2 and b_1. Then $d(y) \geq 6$ and $d_{N(v)}(y) \geq 3$. Suppose then that y is adjacent to exactly one of r_1 or r_2. If either yr_1 or yr_2 is blue, then as y has no other neighbors, it is not possible for y and x_b to have a common blue neighbor without creating a blue triangle. The only remaining possibility is that yr_2 is red, since if yr_1 was red, xyr_1 would be a red triangle. However, then the only blue neighbor of y is b_1, and again it is not possible for y and x_b to have a common blue neighbor. We may therefore conclude that $d(y) \geq 6$ and $d_{N(v)}(y) \geq 3$.

We now show that x_by is in G, so that (3) holds. Suppose otherwise, so that x_by is not in G. Since we cannot add x_by to G_{blue}, there must be some vertex w that is blue-adjacent to each of x_b and y. Furthermore, w cannot be in $N(v)$, as then w would be one of r_1, r_2 or b_1, creating the blue triangles $r_1b_2x_b, r_2x_bx$ or b_1x_bx, respectively. Thus w is in H, implying by Fact 1 that w must be blue-adjacent to either b_1 or b_2, which would create blue K_3 with y or x_b, respectively. Thus, the claim holds under the assumption that r_1b_1 is not in G_{blue}.

Suppose that xr_1 is red and xb_1 and xr_2 are blue, but now we also assume that r_1b_1 is in G_{blue}. We cannot recolor xb_1 red, so there must be a vertex y that is red-adjacent to both x and b_1. Since r_1 is the only red neighbor of x in $N(v)$, and r_1b_1 is blue, y must be in H. Also, since yb_1 is red, yb_2 must be blue.

As in the previous cases, if we enumerate the possible choices of y and sequentially recolor, we may assume that we have selected y such that xy cannot be recolored blue. This implies the existence of a vertex y' that is blue-adjacent to both x and y. Note that $y' \notin H$, since that would imply either $y'b_1$ or $y'b_2$ is in G_{blue}, each of which would lie in a blue K_3. Therefore y' is in $N(v)$, specifically $y' = r_2$, so that yr_2 must be blue. Certainly we cannot recolor xr_2 red, as this would destroy the blue common neighbor of x and y forced above. Hence, there must exist a vertex y'' with red edges to both x and r_2. Note that y'' cannot be y or any vertex in $N(v)$.

The edge xb_2 cannot be added to G_{blue}, so there must exist a blue common neighbor of x and b_2. Since $d(x) < 6$ by assumption, this common neighbor must be either b_1 or r_2. Either case results in a blue triangle (b_2b_1v or r_2b_2y respectively), a contradiction.

Case 2: Suppose x has a wasted red edge.

Specifically, assume that xb_1 and xr_1 are red and xb_2 is blue. Again, we first suppose that the edge b_2r_1 is not in G_{blue} (so it is either missing or red). We cannot recolor xr_1 blue so there must exist a vertex $y \in H$ that is blue-adjacent to both x and r_1. By Fact 1, y must be blue-adjacent to b_1. Since vr_1 cannot be recolored blue and we have assumed that b_2r_1 is not in G_{blue}, there must exist a blue edge from b_1 to r_1. This edge forms a blue triangle with y, a contradiction.

Hence, we may assume b_2r_1 is blue. The edge b_2x cannot be recolored red, so there must exist a vertex y_r that is red-adjacent to both x and b_2. Suppose first that $y_r \in H$. By Fact 1, y_rb_1 must be blue. If we again let $y_1, y_2, \ldots$ enumerate the possible choices for y_r and sequentially recolor the edges xy_r blue, then the fact we cannot recolor all such edges again allows us to select y_r so that xy_r cannot be recolored blue. However, this implies that there is some vertex in H that is blue-adjacent to both x and y_r. Such a vertex cannot be blue-adjacent to either of b_1 or b_2, contradicting Fact 1 and completing the claim.

If, instead, y_r is not in H, then $y_r = b_1$. We cannot recolor vb_2 red, so b_2 must be red-adjacent to some red neighbor of v, say r_2. Also, adding r_2x to G in blue must create a blue K_3, so there must be a vertex $y' \in H$ that is blue-adjacent to both r_2 and x. To avoid creating a blue triangle, $y'b_1$ must be blue.

Now, for some choice of y', $y'r_2$ cannot be recolored red so these two vertices have a red common neighbor (possibly b_2). Similarly, $y'x$ cannot be recolored red so y' and x must have a common red neighbor (possibly r_1). Since r_2x is not in G and the addition of r_2x in red must therefore create a red triangle, there must be another vertex y'' that is red-adjacent to both r_2 and x. Since y'' cannot be r_1 or b_1 without creating a red K_3, $y'' \in H$.

Since $d(x) = 5$, this must account for all of the neighbors of x. We recolor r_2v blue since r_2 has no blue edges to b_1 or b_2. However, we still cannot recolor vb_2 red, so b_2r_3 must be in G_{red}. We have recolored vr_2

blue, but we cannot recolor both vr_2 and vr_3 blue, so we must have either r_2r_3 or r_3b_1 in G_{blue}. Finally, as r_3x is not in $E(G)$, r_3 and x must have a common blue neighbor. Note however that x's blue neighbors are b_2, which is red adjacent to r_3 and y' which if blue-adjacent to r_3 would form a blue triangle. □

We reach a contradiction, and complete the proof of Theorem 2 by carefully counting the edges of G. Partition H into the following sets

- $N_4 = \{x \in H : |N(x) \cap N(v)| \geq 4\}$
- $N_3 = \{x \in H : |N(x) \cap N(v)| = 3 \text{ and } d(x) = 5\}$
- $N_3^* = \{x \in H : |N(x) \cap N(v)| = 3 \text{ and } d(x) \geq 6\}$.

We require one final claim prior to our final count.

Claim 8 *If $x \in H$ has no wasted edges, then either $x \in N_3^* \cup N_4$ or $N_H(x) \cap (N_3^* \cup N_4) \neq \emptyset$.*

Proof: We may assume $x \in N_3$ as otherwise we are done. Suppose first that x has no wasted edges and is blue-adjacent to b_1 and b_2 in $N_{blue}(v)$ and red-adjacent to r_1 in $N_{red}(v)$. We cannot recolor vr_1 blue, so r_1 must be blue adjacent to one of b_1 or b_2, say b_2. We cannot recolor xb_2 red, so b_2 and x must have a common red neighbor y_r. Since r_1b_2 is blue, this neighbor must be in H and therefore has a wasted edge to $N(v)$. Then, by Claim 7, either $y_r \in (N_3^* \cup N_4)$ or there is a vertex $y_r' \in N_H(y_r)$ that has degree at least six and is adjacent to x. This suffices to demonstrate the claim.

Next, assume that x is blue-adjacent to b_1 in $N_{blue}(v)$ and red-adjacent to r_1 and r_2 in $N_{red}(v)$. We cannot recolor r_1x blue so if r_1b_1 is not in G_{blue} there is some vertex y in H that is blue-adjacent to both r_1 and x. As above, the desired conclusion would then follow from Claim 7, so we assume that r_1b_1 and, symmetrically, r_2b_1 are both in G_{blue}. We cannot recolor b_1x red, so there must exist a vertex y in H that is red-adjacent to both b_1 and x. Once again, Claim 7 yields the desired conclusion. □

Since every vertex in H has at least three edges to $N(v)$ and $\delta(G) = 5$, we get that

$$\sum_{v \in G} d(v) \geq 3(n-6) + 5(n-6).$$

Each vertex in N_3^* and N_4 increases this sum by at least one, and we may improve this bound on $\sum d(v)$ as follows.

$$\sum_{v\in G} d(v) \geq 3(n-6) + 5(n-6) + |N_3^* \cup N_4| + \sum_{y\in N_3^*\cup N_4} (d(x)-6).$$

Let $\Theta := |N_3^* \cup N_4| + \sum_{y\in N_3^*\cup N_4}(d(x)-6)$. We claim $\Theta \geq |H|/4$.

If $|N_3^* \cup N_4| \geq |N_3|/3$, then $\Theta \geq |N_3^* \cup N_4| \geq |H|/4$ since $V(H) = N_3 \cup N_3^* \cup N_4$. Suppose then, that $|N_3^* \cup N_4| < |N_3|/3$. In this case, we have $|N_3| \geq 3|H|/4$ and $|N_3^* \cup N_4| \leq |H|/4$. Since each vertex $x \in N_3$ is adjacent to at least one vertex in $N_3^* \cup N_4$, then the number of edges from N_3 to $N_3^* \cup N_4$ is at least $|N_3|$.

Since each $y \in N_3^* \cup N_4$ has at least 3 neighbors in $N(v)$, we have

$$\begin{aligned}
\Theta &= |N_3^* \cup N_4| + \sum_{y\in N_3^*\cup N_4} (d(y)-6) \\
&\geq |N_3^* \cup N_4| + (|N_3| - 3|N_3^* \cup N_4|) \\
&\geq |N_3| - 2|N_3^* \cup N_4| \geq \frac{3}{4}|H| - \frac{2}{4}|H| = |H|/4.
\end{aligned}$$

Thus,

$$\sum_{v\in G} d(v) \geq 3(n-6) + 5(n-6) + \frac{|H|}{4}.$$

We can augment this sum slightly by counting those edges entirely within $N[v]$. We cannot recolor any blue edge incident to v red and, also, we cannot recolor any two red edges incident to v blue. Hence there must be at least four edges completely within $N(v)$ and thus 9 edges completely within $N[v]$. Hence

$$\sum_{v\in G} d(v) \geq 3(n-6) + 5(n-6) + \frac{|H|}{4} + 18,$$

a contradiction for $n \geq 46$. This completes the proof of Theorem 2. □

3. $sat(n, \mathcal{R}_{\min}(K_t, T_m))$

In this section we determine $sat(n, \mathcal{R}_{\min}(K_3, P_3))$ for n at least 11, as a contrast to Theorem 2. First, we recall a classic result of Chvátal [3], which states that if T_m is any tree of order m then $r(K_t, T_m) = (t-1)(m-1)+1$. If we let color one be "red" and color two be "blue", then the lower bound arises from consideration of $(t-1)K_{m-1}$ with every edge colored blue and each edge between the blue cliques colored red. It is well-known that this is the unique edge-coloring of $K_{(t-1)(m-1)}$ with no red K_t and no blue T_m.

Examining the sharpness examples for Theorem 2 and Conjecture 1, it seems reasonable that the correct value of $sat(n, \mathcal{R}_{\min}(K_t, T_m))$ may arise from overlapping copies of $K_{(t-1)(m-1)}$ and demonstrating an appropriate coloring. In particular, we obtain the following upper bound.

Proposition 1 *Let t, m and n be positive integers and let T be a tree of order m. Then $sat(n, \mathcal{R}_{\min}(K_t, T_m))$ is at most*

$$n(t-2)(m-1)-(t-2)^2(m-1)^2+\binom{(t-2)(m-1)}{2}+\left\lfloor\frac{n}{m-1}\right\rfloor\binom{m-1}{2}+\binom{r}{2},$$

where $r \equiv n \pmod{m-1}$.

Proof: Let $H_1 = K_{(m-1)(t-2)}$ and let $H_2 = \left\lfloor\frac{n}{m-1}\right\rfloor K_{m-1} \cup K_r$ and consider $H = H_1 \vee H_2$. Color each edge in H_2 blue, and partition the vertices of H_1 into $t-2$ sets of $m-1$ vertices. Color the cliques induced by each of these sets blue and then color the remaining edges in $H_1 \vee H_2$ red. This coloring contains no red K_t and no blue tree of order m.

We now wish to show that H is $\mathcal{R}_{\min}(K_t, T_m)$-saturated by demonstrating that the coloring of H described above is the unique red/blue coloring of $E(H)$ with no red K_t and no blue T_m. Each copy of K_{m-1} in H_2 is joined to H_1, forming a copy of $K_{(m-1)(t-1)}$. The uniqueness of Chvátal's coloring assures that in any red/blue edge coloring of H that contains no red K_t and no blue T_m, each of these copies must contain a blue $(t-1)K_{m-1}$ with all other edges red. Consequently, the coloring of each K_{m-1} in H_2 must be identical. But then, since every vertex of H_1 lies in a blue copy of K_{m-1}, none of these vertices can be blue-adjacent to two components of H_2, as then this coloring of H would contain a blue K_{m-1} with a pendant edge, and hence a blue copy of T_m. We therefore conclude that no vertex in H_1 lies in a blue K_{m-1} with any vertex from H_2. This implies that each copy of K_{m-1} in H_2 must be colored blue and that every edge from H_1 to H_2 must be red.

We claim next that the component of order r in H_2 must have every edge colored blue. However, since H_1 contains a red copy of K_{t-2} and every edge from H_1 to H_2 is red, a red edge in this K_r would form a red K_t in this coloring of H, a contradiction.

It remains to show that the addition of any edge, red or blue, to this coloring of H results in either a red K_t or a blue T_m Note that the only edges in $\overline{H}$ connect vertices in H_2, so assume that x and y are nonadjacent vertices in H. If the edge xy is added in blue, then without loss of generality x lies in a blue copy of K_{m-1} that does not contain y. This blue complete graph together with the blue edge xy necessarily contains a blue copy of T_m. That the addition of xy in red necessarily creates a red K_t follows from the observation that H_1 contains a red copy of K_{t-2} in which every vertex is connected to x and y by a red edge. □

For instance, if $t = m = 3$, Chvátal's coloring is a red C_4 with a blue matching and in Figure 2 we give a $\mathcal{R}_{\min}(K_3, P_3)$-saturated graph arising from this coloring of K_4.

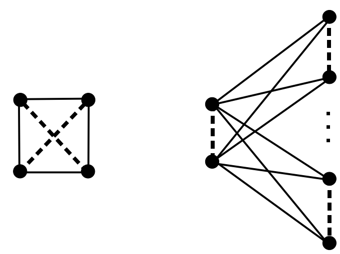

Figure 2: A coloring of K_4 with no red K_3 or blue P_3 that gives rise to a $\mathcal{R}_{\min}(K_3, P_3)$-saturated graph.

This graph has $\lfloor \frac{5n}{2} \rfloor - 4$ edges and seems like a good candidate for a $\mathcal{R}_{\min}(K_3, P_3)$-saturated graph of minimum size. In fact, we can do slightly better.

Theorem 4 *For $n \geq 11$,*

$$sat(n, \mathcal{R}_{\min}(K_3, P_3)) = \left\lfloor \frac{5n}{2} \right\rfloor - 5.$$

Prior to proving Theorem 4, we require the following result of Barefoot, et al.[1].

Theorem 5 *Let $n \geq 5$ be an integer and let G be a K_3-saturated graph of order n. Then either G is a complete bipartite graph or $2n - 5 \leq e(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$.*

Let G be a graph and let v be a vertex in G. We *inflate* the vertex v in G by replacing v with an independent set of vertices and connecting each of these new vertices to the neighbors of v in G. The next lemma follows directly from the proof of Theorem 5 given in [1], and is also implied by the proof of Corollary 3.1 in [4], so we omit the proof here.

Lemma 1 *Let G be a 2-connected K_3-saturated graph of order n with exactly $2n-5$ edges. Then G can be obtained by inflating two nonadjacent vertices of C_5.*

We are now ready to prove Theorem 4.

Proof: As above, suppose we are trying to assure either a red K_3 or a blue P_3 and let G be a $\mathcal{R}_{\min}(K_3, P_3)$-saturated graph. If there exists an edge $e \in G$ which is in three different triangles of G, then the only way to edge-color these triangles without a red K_3 or a blue P_3 is to color e blue and the remaining edges red. Hence, we get the following fact which will help us establish our lower bound on $sat(n, \mathcal{R}_{\min}(K_3, P_3))$.

Fact 2 *In any red/blue edge coloring of a $\mathcal{R}_{\min}(K_3, P_3)$-saturated graph G with no red K_3or blue P_3, any edge e lying in three or more triangles must be colored blue.*

To establish the upper bound in Theorem 4, consider a copy of C_5 with vertices $v_1, \ldots, v_5$ appearing in that order on the cycle. Inflate v_1 to a set V_1 of at least three vertices to obtain a graph of order $n \geq 7$ and color all $2n-5$ edges of this graph red. Next we add a matching, in blue, that consists of the edge v_2v_5 and a maximum matching M amongst the remaining $n-2$ vertices that does not include the edge v_3v_4, as this edge is already present. Call this graph G_0 and note that the coloring given contains no red K_3 and no blue P_3.

It remains to show that G_0 is $\mathcal{R}_{\min}(K_3, P_3)$-saturated, so consider a red/blue coloring of $E(G_0)$ that contains no red K_3 and no blue P_3. Note that the edge v_2v_5 lies in a triangle with each vertex in V_1, so by Fact 2, it must be colored blue in any coloring with no red K_3 or blue P_3. This implies that all of the other edges incident to v_2 and v_5 must be colored red. Now we note that each edge in $M - \{v_2v_5\}$ is of the form v_3x, v_4x or xy for vertices x, y in V_1, which forces each of these edges to be blue, and hence forces the remaining edges in $G_0 - M$ to be red. Thus we have forced the coloring of G_0 described above, in which the addition of any edge, in red or blue, forces a red K_3 or a blue P_3.

To establish the lower bound, let G be a $\mathcal{R}_{\min}(K_3, P_3)$-saturated graph on n vertices with the minimum number of edges. Furthermore, consider a coloring of G containing no red K_3 and no blue P_3 having the maximum number of red edges. Notice that if two vertices u and v each have no blue neighbor, then they must be red-adjacent. Hence, the next fact follows immediately.

Fact 3 *There is no set T of at least three vertices in G each with no blue neighbors.*

As above, let G_r and G_b denote the graph induced by the red edges and the blue edges in G, respectively.

Claim 9 *The graph G_r is 2-connected.*

Proof: Note first that G_r is connected, as the addition of a red edge between two components R_1 and R_2 in G_r could not create a triangle, and hence every edge between R_1 and R_2 would have to be in G_b. This would imply that every edge connecting a vertex in R_1 and a vertex in R_2 would be in G_b, creating a blue P_3.

Suppose then that G_r had connectivity one, and let v be a cut-vertex in G_r. Let C_1 and C_2 be components of $G_r - v$ and suppose that there is a vertex w in C_1 that is not adjacent to v. Let x be any vertex in C_2. Then the edge wx can either be added in red or changed from blue to red, contradicting the choice of G in each case.

Hence, we may suppose v is adjacent to all of $G_r - v$ implying that G_r must be a star centered at v. Examining G, there must be a blue matching amongst the vertices of $G - v$. Suppose that xy is an edge of this matching. If we recolor vx blue and xy red, we may then add an edge from x to $G - \{v, x\}$ in red, contradicting the assumption that G is saturated. $\square$

The remainder of the proof is broken into cases based on the parity of n.

Case 1 *n is odd.*

First, we claim that G_r is maximal triangle-free. If not, we could either add a red edge to G or recolor a blue edge of G red. Either way, this contradicts our choice of G. Since G_r is 2-connected, Theorem 5 yields that G_r has at least $2n - 5$ edges. By Fact 3, there must be at most one vertex (since n is odd) with no incident blue edge. Hence, $e(G) \geq 2n - 5 + \lfloor \frac{n}{2} \rfloor = \lfloor \frac{5n}{2} \rfloor - 5$, completing this case.

Case 2 *n is even.*

The coloring of G was chosen so that the red graph has as many edges as possible, so if there are any edges which could be colored either red or blue without creating a monochromatic K_3 or P_3, they will be red. Consequently we may again assume that the red graph is K_3-saturated but also, by Fact 3, that there are at most two vertices with no blue neighbor.

If the blue graph is a perfect matching, then by Lemma 1 and the fact that G_{red} is 2-connected, $e(G) \geq 2n - 5 + \frac{n}{2} = \frac{5n}{2} - 5$, completing the result. Thus, suppose that there is a pair of vertices which are not covered by the blue matching. These vertices must be joined by an edge e in red. We therefore assume, since G_{red} is 2-connected and K_3-saturated, that $e(G_{red}) = 2n - 5$ and furthermore that there are exactly $\frac{n}{2} - 1$ blue edges. Note that e can be recolored blue without creating a blue P_3.

By Lemma 1, since $n \geq 11$, G_r must be a copy of C_5 with two nonadjacent vertices inflated. Consider a C_5 with vertices $v_1, v_2, \ldots, v_5$. Let v_1 and v_3 be the inflated vertices (as in the structure provided by Lemma 1) and let V_1 and V_3 be the corresponding independent sets. The remainder of the proof is broken into cases based on the location of e in this structure.

If $e = v_4v_5$, then since v_2 is adjacent to every vertex in V_1 and V_3, v_2 has no blue neighbor, a contradiction. Suppose then that $e = v_4a_3$ for some vertex $a_3 \in V_3$ (or symmetrically $e = v_5a_1$ for some $a_1 \in V_1$). We may then color e blue and add the edge v_5a_3 in red without creating a red triangle. This contradicts the assumption that G was $\mathcal{R}_{\min}(K_3, P_3)$-saturated.

Finally suppose that $e = v_2a_3$ for some $a_3 \in V_3$ (or symmetrically $e = v_2a_1$ for some $a_1 \in V_1$). Since G_{blue} is a matching saturating all of $V(G) - \{v_2, a_3\}$, v_5 must have a blue neighbor a_3' in V_3. Likewise v_4 must have a blue neighbor a_1'. We may then recolor v_5a_3' and v_4a_1' red, v_4v_5 blue, and add the edge $a_3'a_1'$ in blue. This contradicts the assumption that G was $\mathcal{R}_{\min}(K_3, P_3)$-saturated and completes the proof. $\square$

4. Conclusion and Open Problems

In this paper, we have verified the first non-trivial case of Conjecture 1. It would be of interest to determine non-trivial lower bounds on $sat(n, \mathcal{R}_{\min}(K_{k_1}, \ldots, K_{k_t}))$. For instance, if Conjecture 1 were to hold, a classic result of Spencer [10] would imply that

$$sat(n, \mathcal{R}_{\min}(K_k, K_k)) \geq (1 + o(1))\frac{\sqrt{2}}{e}k2^{\frac{k}{2}}n,$$

where the $o(1)$ term is with respect to k. At this time, we are only able to show the following.

Claim 10

$$sat(n, \mathcal{R}_{\min}(K_k, K_k)) \geq \frac{(k-1)^2 - 1}{2} n.$$

PROOF: It is readily seen that a graph H satisfies $H \to (K_k, K_k)$ if and only if H contains a K_k-minimal subgraph H'. Let G be a $R_{min}(K_k, K_k)$-saturated graph. Then, for any edge $e \in \overline{G}$, $G + e$ contains a K_k-minimal subgraph. A result of Burr, Erdős and Lovász [2], reproved recently by Fox and Lin [6], states that the minimum degree of a K_k-minimal subgraph is at least $(k-1)^2$. This implies that the minimum degree of G is at least $(k-1)^2 - 1$, and the result follows. □

In addition to Conjecture 1, one may investigate $sat(n, \mathcal{R}_{\min}(G, H))$ for other pairs of graphs. As a starting point, we conjecture that $sat(n, \mathcal{R}_{\min}(K_t, T_m))$ is the same asymptotically as the bound given in Proposition 1.

Finally, as mentioned above, Galluccio, Simonovits and Simonyi have obtained a number of results on (not necessarily minimal) $\mathcal{R}_{\min}(K_3, K_3)$-saturated graphs in [7]. The interested reader may wish to investigate the wealth of interesting and challenging conjectures and open problems posed in that paper.

References

[1] C. Barefoot, K. Casey, D. Fisher, K. Franaugh and F. Harary, Size of maximal triangle-free graphs and minimal graphs of diameter 2, *Discrete Math.* **138** (1995), 93–99.

[2] S. Burr, P. Erdős, L. Lovász, On graphs of Ramsey type. *Ars Combin.* **1** (1976), 167–190.

[3] V. Chvátal, Tree-complete graph Ramsey numbers, *J. Graph Theory* **1** (1977), 93.

[4] D. Duffus and D. Hanson, Minimal k-saturated and color critical graphs of prescribed minimum degree, *J. Graph Theory* **10** (1986), 55–67.

[5] P. Erdős, A. Hajnal and J.W. Moon, A problem in graph theory. *Amer. Math. Monthly* **71** (1964), 1107–1110.

[6] J. Fox and K. Lin, The minimum degree of Ramsey-minimal graphs, *J. Graph Theory* **54** (2007), 167–177.

[7] A. Galluccio, M. Simonovits, G. Simnoyi, On the structure of co-critical graphs, *Graph theory, combinatorics, and algorithms, Vol. 1,2 (Kalamazoo, MI, 1992)*, 1053–1071, Wiley-Intersci. Publ., Wiley, New York, 1995.

[8] D. Hanson and B. Toft, Edge-colored saturated graphs, *J. Graph Theory* **11** (1987), 191–196.

[9] O. Pikhurko, Results and open problems on minimum saturated graphs, *Ars Combin.* **72** (2004) 111–127.

[10] J. Spencer, Ramsey's theorem - a new lower bound, *J. Combin. Theory, Ser. A* **18** (1975), 108–115.

Guantao Chen
Georgia State University, Atlanta, GA 30302, USA,
E-mail address: gchen@gsu.edu

Michael Ferrara
University of Colorado Denver, Denver, CO 80217, USA,
E-mail address: michael.ferrara@ucdenver.edu

Ronald J. Gould
Emory University, Atlanta, GA 30322, USA
E-mail address: rg@mathcs.emory.edu

Colton Magnant
Georgia Southern University, Statesboro, GA 30460, USA
E-mail address: cmagnant@georgiasouthern.edu

John Schmitt
Middlebury College, Middlebury, VT 05753, USA,
E-mail address: jschmitt@middlebury.edu