POTENTIALLY H-BIGRAPHIC SEQUENCES

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ABSTRACT. We extend the notion of a potentially H-graphic sequence as follows. Let A and B be nonnegative integer sequences. The sequence pair $S = (A, B)$ is said to be *bigraphic* if there is some bipartite graph $G = (X \cup Y, E)$ such that A and B are the degrees of the vertices in X and Y, respectively. If S is a bigraphic pair, let $\sigma(S)$ denote the sum of the terms in A.

Given a bigraphic pair S, and a fixed bipartite graph H, we say that S is *potentially H-bigraphic* if there is some realization of S containing H as a subgraph. We define $\sigma(H, m, n)$ to be the minimum integer k such that every bigraphic pair $S = (A, B)$ with $|A| = m, |B| = n$ and $\sigma(S) \geq k$ is potentially H-bigraphic. In this paper, we determine $\sigma(K_{s,t}, m, n)$, $\sigma(P_t, m, n)$ and $\sigma(C_{2t}, m, n)$.

1. INTRODUCTION

Let $S = (A, B) = (a_1, \ldots, a_m; b_1, \ldots, b_n)$ be a pair of positive integer sequences. We say that S is a *bigraphic pair* if there exists some simple bipartite graph G with partite sets $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ such that the degree of x_i is a_i and the degree of y_j is b_j. In this case, we say that G is a bigraphic realization of S. In this paper, as the bipartite context is clear, we will simply call G a realization of S. One easy method to determine if a given sequence pair is bigraphic is the Gale-Ryser condition [3, 11]. Given a bipartite graph H and a bigraphic pair S, we say that S is *potentially H-bigraphic* if there is some realization of S that contains H as a subgraph. This is a weakening of the Zarankiewicz problem [12], which is the bipartite analogue to determining the extremal function for arbitrary subgraphs. This seemingly innocent variant to the classical Turán problem has proven to be much more challenging over time. A good discussion of the problem and its rich history can be found in [1].

Given a bigraphic sequence pair $S = (A, B)$, let $\sigma(S)$ denote the sum of the terms in either A or B (which are necessarily equal). For a given bipartite graph H, let $\sigma(H, m, n)$ denote the minimum integer k such that any bigraphic pair $S = (A, B)$ with $|A| = m, |B| = n$ and $\sigma(S) \geq k$ is potentially H-bigraphic. This is a natural

Key words and phrases. graphic sequence, bipartite, potentially graphic, 2000 Mathematics Subject Classification: 05C07.

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extension of the notion of a potentially H-graphic sequence, which has been widely studied.

In this paper, we will determine $\sigma(H, m, n)$ for several graphs H. In Section 2, we determine $\sigma(K_{s,t}, m, n)$, where $K_{s,t}$ is the complete bipartite graph with vertex sets of size s and t. In Section 3 we find $\sigma(P_t, m, n)$, where P_t is the path on t vertices. Finally, in Section 4, we use the two previous results to determine $\sigma(C_{2t}, m, n)$ for even cycles C_{2t}.

The following useful lemma is an extension of a result found in [4].

Lemma 1.1. *Let S be a bigraphic pair with realization $G = (X \cup Y, E)$ having partite sets X and Y. Let $H = (X' \cup Y', E')$ be a subgraph of G such that X' and Y' are contained in X and Y, respectively. Then there exists a realization $G_1 = (X \cup Y, E_1)$ of S containing H as a subgraph such that X' and Y' lie on the vertices of highest degree in X and Y, respectively.*

Proof. Let $G = G(X \cup Y, E)$ be a realization of bigraphic sequence S containing a graph H as a subgraph, such that $\{u, v\} \subset X$, (or $\{u, v\} \subset Y$,) $u \notin V(H)$, $v \in V(H)$, and $\deg_G(u) \geq \deg_G(v)$. Let $T = N_H(v) \setminus N_G(u, v)$ be the neighbors of v in H that are not neighbors of u. Since $|N_G(u)| \geq |N_G(v)|$, we have that

$$|N_G(u) \setminus N_G(u, v)| \geq |N_G(v) \setminus N_G(u, v)| \geq |T|,$$

thus there exists subset T' of $N_G(u) \setminus N_G(u, v)$ of size $|T|$. Let $G' = G'(X \cup Y, E')$ where

$$E' = E \setminus \big(E(u + T') \cup E(v + T)\big) \cup \big(E(u + T) \cup E(v + T')\big).$$

Then G' is a realization of S containing a copy of H with vertex u in place of vertex v. The lemma follows. $\square$

Throughout this paper, we will assume each sequence in a given sequence pair is nonincreasing. We will also often use exponential notation for a degree sequece. That is, we will write $(a_1^{\alpha_1}, \ldots, a_r^{\alpha_r}; b_1^{\beta_1}, \ldots, b_s^{\beta_s})$ to denote the sequence pair

$$(a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_r, \ldots, a_r; b_1, \ldots, b_1, b_2, \ldots, b_2, \ldots, b_s, \ldots, b_s)$$

in which a_i and b_j occur α_i and β_j times respectively.

2. Complete Bipartite Graphs

In this section, we determine $\sigma(K_{s,t}, m, n)$. The problem of determining when a graphic sequence contains a copy of $K_{s,t}$ has been studied, and the interested reader may wish to compare the corresponding results, found in [8] and [9]. In the bipartite setting, determining $\sigma(K_{s,t}, m, n)$ might be considered analagous to determining when a graphic sequence has a realization containing a copy of K_t, as in [2], [4], [6], and [7].

Theorem 2.1. *For all $1 \leq s \leq t$, there exists m_0 such that for $n \geq m \geq m_0$ the following holds.*

$$\sigma(K_{s,t}, m, n) = n(s-1) + m(t-1) - (t-1)(s-1) + 1.$$

Proof. We begin by exhibiting a bigraphic pair S with $\sigma(S) = n(s-1) + m(t-1) - (t-1)(s-1)$ which is not potentially $K_{s,t}$-bigraphic. Consider the sequence pair

$$S = (n^{s-1}, (t-1)^{m-s+1}; m^{s-1}, (t-1)^{m-s+1}, (s-1)^{n-m}).$$

This sequence is bigraphic, and neither partite set in any realization of S has s vertices of degree t. Hence S is not potentially $K_{s,t}$-graphic.

Moving forward, let S be a bigraphic pair with $\sigma(S)$ at least $n(s-1)+m(t-1)-(t-1)(s-1)+1$. Let G be a realization of S with partite sets X and Y, with $|X|=n$ and $|Y|=m$. Let X_t be the set of t highest degree vertices of X, and Y_s be the set of s highest degree vertices of Y. Assume that G is a realization of S that maximizes the number of edges between X_t and Y_s. If the graph on $X_t \cup Y_s$ is $K_{s,t}$ we are done, so assume otherwise. Let x and y be nonadjacent members of X_t and Y_s, and let $H_X = X_t \setminus \{x\}$ and $H_Y = Y_s \setminus \{y\}$.

Let A denote $N(y) \setminus H_X$ and let B denote $N(x) \setminus H_Y$. Note that neither A nor B is empty, as it is straightfoward to show that x and y have degrees at least s and t, respectively.

Claim 2.2. *Let a and b lie in A and B respectively. Then ab is an edge of G.*

Proof. Assume otherwise, and exchange the edges ya and xb for the nonedges ab and xy. This preserves the degree sequence of G, but contradicts our assumption that G had the maximum number of edges between X_t and Y_s among all realizations of S. □

Claim 2.2 implies that the subgraph of G induced by A and B is a complete bipartite graph.

Claim 2.3. *For each b in B there exists a vertex h_x in H_X such that b is not adjacent to h_x. Similarily, for each a in A there exists a vertex h_y in H_Y such that a is not adjacent to h_y.*

Proof. We prove the first statement. The proof of the second is similar. Assume the first statement is false. Then, as b is adjacent to x,

$$d(b) \geq |H_X| + |A| + 1 > d(y).$$

This contradicts the fact that y is one of the s highest degree vertices in Y. □

Claim 2.3 immediately implies the following two claims.

Claim 2.4. *Let b and h_x be nonadjacent vertices in B and H_X respectively. Then for all a in A and all v in $N(h_x) \setminus (Y_s \cup B)$, av is an edge of G.*

The analagous statement about nonadjacent a and h_y in A and H_Y respectively, is also true.

Proof. Again, we prove just the first statement. Assume it is false, i.e., that there is some $a \in A$ and some $v \in N(h_x) \setminus (Y_t \cup B)$ that are not adjacent. Then we could exchange the edges ay, bx and $h_x v$ for the nonedges $h_x b, av$ and xy. This is, again, a contradiction to our choice of G. □

This allows us to bound the number of vertices in A and B as follows.

Claim 2.5. *Let A and B be as defined above. Then both A and B contain at most*

$$(s-1)(t-1)$$

vertices.

Proof. We prove $|B| \leq (s-1)(t-1)$. The proof for $|A|$ is similar. Assume that $|B| > (s-1)(t-1)$. By Claim 2.3 and the pigeonhole principle there must be some h_x in H_X that is non-adjacent to at least s vertices in B. Its neighborhood is

$$N(h_x) = [N(h_x) \setminus (Y_s \cup B)] \cup [N(h_x) \cap Y_s] \cup [N(h_x) \cap B],$$

so we have that

$$d(h_x) \leq |N(h_x) \setminus (Y_s \cup B)| + s + (|B| - s) = |N(h_x) \setminus (Y_s \cup B)| + |B|.$$

On the other hand, for any vertex a in A, Claim 2.4 and the comment following Claim 2.2 implies that the neighborhood of a contains

$$[N(h_x) \setminus (Y_s \cup B)] \cup B \cup \{y\},$$

so we have

$$d(a) \geq |N(h_x) \setminus (Y_s \cup B)| + |B| + 1 > d(h_x). \tag{1}$$

This contradicts the fact that h_x is in X_t. □

Claim 2.6. *Let h_x and h_y be as given above. Then*

$$d(h_x) < 2s + |B| \qquad \text{and} \qquad d(h_y) < 2t + |A|.$$

Proof. Assume $d(h_x) \geq |B| + 2s$. Then by equation (1), we have for any a in A that

$$d(a) \geq [d(h_x) - |(Y_s \cup B)|] + |B| + 1 > |B| + s > d(x).$$

This contradicts the assumption that $d(x) \geq d(a)$. The proof for h_y is similar. □

Now since both $d(x)$ and $d(h_x)$ are bounded by $2s + |B|$, and they are both in X_t the number of edges in G that are incident to vertices of either X_t or A is at most

$$(t-2)m + (2s + |B|)(|A| + 2).$$

Similarly the number of edges incident to Y_s or B is at most

$$(s-2)n + (2t + |A|)(|B| + 2).$$

By Claim 2.5, this accounts for at most

$$\begin{aligned}(t-2)m + (s-2)n \;&+\; (2s + (s-1)(t-1))((s-1)(t-1) + 2)\\ &+\; (2t + (s-1)(t-1))((s-1)(t-1) + 2),\end{aligned}$$

which is less than $(t-2)m + (s-2)n + 6s^2t^2$, edges from G. Taking m and n larger than $4s^2t^2$ this is strictly less than $\sigma(S)$.

Furthermore, A has at most $|A|d(h_x)$ neighbors, which by Claims 2.5 and 2.6, is at most

$$(2s + (s-1)(t-1))\,(s-1)(t-1) < 3s^2t^2.$$

Each of these vertices which is outside of Y_s has at most $d(h_y) < 2t + (s-1)(t-1)$ neighbors. Thus at most $9s^4t^3$ vertices in X have neighbors outside of Y_s which are adjacent to vertices of A. Assuming that $m_0 = 9s^4t^4$, together $n > m_0 > 9s^4t^3$ and $m > m_0 > 9t^4s^3$, ensure that there exists some edge $e = x'y'$, with $x' \in X - X_t - A$ and $y' \in Y - Y_t - B$, and vertices a and b in A and B respectively, such that $x'b$ and $y'a$ are not edges in G. We can then exchange the edges ab and e for the non-edges $x'a$ and $y'b$, contradicting Claim 2.2, and completing the proof. □

We note in the proof that the sets A and B induce a complete bipartite graph. Hence at least one of A and B contains at most $t-1$ vertices, and if either contains more than $t-1$ vertices, the other set contains at most $s-1$ vertices. This would be useful if one were interested in finding smaller bounds on the n and m necessary to assure Theorem 2.1.

3. Paths

Recall that P_t denotes the path on t vertices. In this section we determine the quantity $\sigma(P_t, m, n)$. In particular, we prove the following.

Theorem 3.1. *For $t \geq 2$ and integers $n \geq m \geq t+1$,*

$$\sigma(P_{2t+1}, m, n) = \sigma(P_{2t+2}, m, n) = n(t-1) + m - (t-1) + 1.$$

To see that both $\sigma(P_{2t+1}, m, n)$ and $\sigma(P_{2t+2}, m, n)$ are greater than $n(t-1) + m - (t-1)$, consider the sequence pair $S = (m^1, (t-1)^{n-1}; n^{t-1}, 1^{m-t+1})$. This pair has $\sigma(S) = n(t-1) + m - (t-1)$ and has a unique realization, which contains no P_{2t+1}.

The remainder of the section is dedicated to showing that a bigraphic sequence with the above sum has a realization containing a P_{2t+1} and a realization containing a P_{2t+2}. The proof will be by induction on t. The following lemma is sufficient to act as a basis for this induction, and is also of interest for the sake of completeness.

Lemma 3.2. *Let $n \geq m$ be integers. Then,*
 (i) *$\sigma(P_3, m, n) = m + 1$, and*
 (ii) *$\sigma(P_4, m, n) = n + 1$.*

Proof. That $\sigma(P_3, m, n) \geq m+1$ and $\sigma(P_4, m, n) \geq n+1$ is obvious. Equality for statement (i) follows from the fact that with degree sum $m+1$ some vertex in any realization must have degree 2, and hence be the center vertex of a P_3. For statement (ii) observe that with degree sum $n+1$, at least one vertex in each partite set has degree 2 or more. Applying Lemma 1.1 with $H = K_{1,1}$, there exists a realization in which these vertices are adjacent, and hence lie in a P_4. □

The induction follows immediately from the following two lemmas.

Lemma 3.3. *For $t \geq 2$ and integers $n \geq m \geq t+1$, if S is a bigraphic pair with*

$$\sigma(S) \geq n(t-1) + m - (t-1) + 1,$$

and S is potentially P_{2t}-bigraphic, then S is potentially P_{2t+1}-bigraphic.

Lemma 3.4. *For $t \geq 2$ and integers $n \geq m \geq t+1$, if S is a bigraphic pair with*

$$\sigma(S) \geq n(t-1) + m - (t-1) + 1,$$

and S is potentially P_{2t+1}-bigraphic, then S is potentially P_{2t+2}-bigraphic.

To finish the proof of Theorem 3.1 we thus prove Lemmas 3.3 and 3.4.

Proof. (of Lemma 3.3)
Let $S = (A, B) = (a_1 \ldots, a_n; b_1, \ldots, b_m)$ be a bigraphic pair with $\sigma(S) = n(t-1) + m - (t-1) + 1$, and let $G = G(X \cup Y, E)$ be a realization of S, with $|X| = m$ and $|Y| = n$, that contains a P_{2t}. By Lemma 1.1 we may assume that the copy of P_{2t} occurs on vertex sets $X_t := \{x_1, \ldots, x_t\}$ and $Y_t := \{y_1, \ldots, y_t\}$. We must now

show that some realization G' of S contains a P_{2t+1}. We proceed by contradiction and assume that no realization of S, including G, contains a P_{2t+1}.

The following claim allows us to further assume that there is no C_{2t} on $X_t \cup Y_t$.

Claim 3.5. *If the subgraph induced by $X_t \cup Y_t$ contains a cycle C_{2t}, then S is potentially P_{2t+1}-bigraphic.*

Proof. Assume that the subgraph induced by $X_t \cup Y_t$ contains a copy of C_{2t}. If there exists an edge with one endpoint in $X_t \cup Y_t$ and one endpoint outside of this set then we are done. Thus we may assume there exists no such edge. Since $m, n > t$, there exists a pair of vertices x, y in $V(G) - (X_t \cup Y_t)$, and by assumption, each has degree at least 1. We may assume that xy is an edge for $x \in X - X_t$ and $y \in Y - Y_t$ and so for any edge $x'y'$ in the C_{2t}, $x' \nsim y \sim x \nsim y' \sim x'$ is an alternating cycle in G. Removing the edges of this alternating cycle from G and putting the non-edges into G, we arrive at another realization of the same degree sequence, which contains a P_{2t+1}. □

Let the P_{2t} on $X_t \cup Y_t$ be

$$v_1, v_2, \ldots, v_{2t-1}, v_{2t}$$

where vertices with odd index are in X_t and those with even index are in Y_t. Clearly, neither of v_1 and v_{2t} can have neighbors outside of $X_t \cup Y_t$. Moreover, where $d_X = \deg(v_1)$ and $d_Y = \deg(v_{2t})$, the following is also true.

$$d_X + d_Y \leq t \tag{2}$$

Indeed if G contained both of the edges $v_1 v_{2i}$ and $v_{2t} v_{2i-1}$, for any $i = 1, \ldots, t$, then it would contain the $2t$-cycle

$$v_1, v_2, \ldots, v_{2i-1}, v_{2t}, v_{2t-1}, \ldots, v_{2i}, v_1.$$

This would contradict Claim 3.5.

Now the number of edges in G is the number of edges incident to $(Y - Y_t) \cup \{v_{2t}\}$ or $(X - X_t) \cup \{v_1\}$, plus the number between $X_t - \{v_1\}$ and $Y_t - \{v_{2t}\}$. This is at most

$$(m - (t-1))d_X + (n - (t-1))d_Y + (t-1)^2.$$

Since $n \geq m$, and $d_X \geq 1$, this is at most

$$n(d_X + d_Y - 1) + m - (t-1)(d_X + d_Y) + (t-1)^2.$$

Since $n > t$ and $d_X + d_Y \leq t$, this is maximized when $d_X + d_Y = t$, so is at most

$$n(t-1) + m - (t-1).$$

This, however, is one less than $\sigma(S)$, which is a contradiction.

□

Proof. (of Lemma 3.4)

Let S be a bigraphic pair with $\sigma(S) \geq n(t-1) + m - (t-1) + 1$, and let G be a realization of S that contains a P_{2t+1}.

We first consider the case in which the endpoints of the P_{2t+1} occur in X. By Lemma 1.1 we may assume that the copy of P_{2t+1} occurs on vertex sets $X_{t+1} := \{x_1, \ldots, x_{t+1}\}$ and $Y_t := \{y_1, \ldots, y_t\}$. We show that G contains a P_{2t+2}. The proof is, again, by contradiction.

Let e_X denote the number of vertices of X_{t+1} that are the endpoint of some P_{2t+1} on $X_{t+1} \cup Y_t$. Let x be any such endpoint, and observe that

$$e_X \geq deg(x) + 1. \tag{3}$$

Indeed, let $x = v_1, \ldots, v_{2t+1}$ be a P_{2t+1} with $x = v_1$ as an endpoint. For every edge $v_1 v_i$, the following is a P_{2t+1} having v_{2t+1} as an endpoint:

$$v_{i-1}, v_{i-2}, \ldots, v_1, v_i, v_{i+1}, v_{i+2}, \cdots, v_{2t+1}.$$

As v_{2t+1} is also counted by e_X, the inequality holds.

Since each vertex of X_{t+1} which is counted by e_X has degree at most $e_X - 1$, we can bound the number of edges in G by

$$\begin{aligned} & n(t+1-e_X) + (e_X - 1)[m - (t+1-e_X)] \\ = \; & n(t - (e_X - 1)) + m(e_X - 1) - (e_X - 1)(t - (e_X - 1)) \end{aligned}$$

Because $n \geq m$ and $1 \leq e_X - 1 \leq t$, this is maximized when $e_X - 1 = 1$, so is at most

$$n(t-1) + m - (t-1).$$

This is one less than $\sigma(S)$, so completes the proof in the case that the endpoints of the P_{2t+1} are in X.

When the endpoints of the P_{2t+1} are in Y, then analogous arguments allow us to bound the number of edges in G by

$$m(t - (e_Y - 1)) + n(e_Y - 1) - (e_Y - 1)(t - (e_Y - 1)), \tag{4}$$

where e_Y denotes the number of vertices in Y_{t+1} that are endpoints of a P_{2t+1} on $X_t \cup Y_{t+1}$.

Claim 3.6. *We have the following inequality,*

$$1 \leq e_Y - 1 < t.$$

Proof. It is trivial from the definition that $1 \leq e_Y - 1 \leq t$. Assume now that $e_Y - 1 = t$, so all vertices in Y_{t+1} are endpoints of a P_{2t+1}. Then there are no edges from Y_{t+1} to $X - X_t$, or else we have a P_{2t+2}. So every vertex of Y_{t+1} has degree at most t. Since by the degree sum, some vertex of Y_{t+1} must have degree at least t, there is some vertex y of Y_{t+1} that is adjacent to every vertex in X_t. In particular, when

$$y = v_1, v_2, \ldots, v_{2t+1}$$

is the P_{2t+1} with y as endpoint, y is adjacent to v_{2t}.

Now since $m > t+1$, there is a vertex $x_0 \in X - X_t$ which must have some edge. Let y_0 in $Y - Y_{t+1}$ be the other endpoint of this edge. If $y_0 \sim v_{2t}$ then we have the P_{2t+2}

$$v_1, \ldots, v_{2t}, y_0, x_0.$$

If $y_0 \nsim v_{2t}$, then we have the alternating path

$$v_1 \sim v_{2t} \nsim y_0 \sim x_0 \nsim v_1.$$

Removing the edges of this path from G and replacing them with the non-edges, we get a new realization of the same degree sequence which has the P_{2t+2}

$$x_0, v_1, v_2, \ldots, v_{2t}, y_0.$$

This completes the proof of the claim. $\square$

With this claim we have that the $(e_Y - 1)(t - (e_Y - 1))$ of equation (4) is minimized when $e_Y - 1 = 1$ and because $m \leq n$, the positive terms are maximized when $e_Y - 1 = t - 1$, thus it is bounded above by

$$m(t-(t-1)) + n(t-1) - (1)(t-(1)) = n(t-1) + m - (t-1).$$

Again, this is one less than $\sigma(S)$, so is a contradiction.

This completes the proof of the Lemma 3.4 and therefore completes the proof of Theorem 3.1.

□

4. Even Cycles

The minimum degree sum necessary to assure a graphic sequence has a realization containing a copy of C_t was determined in [5], and [10]. Here, we look at the similar problem of determining the minimum sum needed to assure that a bigraphic pair has a realization containing a copy of C_{2t}.

Theorem 4.1. *Given $t \geq 2$, and $n \geq m \geq 2(t+1)$,*

$$\sigma(C_{2t}, m, n) = n(t-1) + m - (t-1) + 1.$$

Proof. The case $t = 2$ follows from Theorem 2.1, so we assume that $t > 2$. The fact that the given value is a lower bound for $\sigma(C_{2t}, m, n)$ is established by the same bigraphic sequence given in Theorem 3.1. We now show that it is also an upper bound.

Let S be a bigraphic pair with $\sigma(S) \geq n(t-1)+m-(t-1)+1$. By Theorem 3.1 we get a realization G of S with a copy of P_{2t+2}. By Lemma 1.1 we may assume that this P_{2t+2} occurs on $X_{t+1} = \{x_1, \ldots, x_{t+1}\}$ and $Y_{t+1} = \{y_1, \ldots, y_{t+1}\}$. The following claim allows us to assume that G contains a C_{2t+2}

Claim 4.2. *Let P be a copy of P_{2t+2} in G. If the endpoints of P are not adjacent, then S is potentially C_{2t}-bigraphic.*

Proof. Let x and y be the endpoints of P and y' and x' be their respective neighbors in P. If x' is adjacent to y' then we have a C_{2t} and are done. Thus we assume $x' \nsim y'$. Now if $x \nsim y$ then $x \nsim y \sim x' \nsim y' \sim x$ is an alternating cycle whose reversal yields a C_{2t} in G. Thus we may assume that x and y are adjacent. □

We therefore make the assumption that G contains a C_{2t+2} on the vertices $v_1, v_2, \ldots, v_{2t+2}$, where the vertices with even index are in X and those with odd index are in Y. The following claim allows us to assume that the C_{2t+2} is induced.

Claim 4.3. *If G contains a cycle C of length $2t+2$ that is not induced, then S is potentially C_{2t}-bigraphic.*

Proof. Assume that C contains a chord, wlog $v_1 \sim v_{2j}$ for some j, $2 \leq j \leq t$. Then we have the P_{2t+2}

$$v_2, v_3, \ldots, v_{2j}, v_1, v_{2t+2}, v_{2t+1}, \ldots, v_{2j+1}$$

with endpoints v_2 and v_{2j+1}. By Claim 4.2, we may assume that these endpoints are adjacent. The same argument applied to the chord $v_2 v_{2j+1}$ shows that v_3 and v_{2j+2} are adjacent as well. However, this implies that G contains the C_{2t} $v_3, v_4, \ldots, v_{2j}, v_1, v_{2t+2}, v_{2t+1}, \ldots, v_{2j+2}, v_3$. This proves the claim.

□

Now let C refer to the induced copy of C_{2t+2} in G and consider its vertices v_1 and v_6.

Claim 4.4. *We may assume that v_1 and v_6 each have degree at least* 3.

Proof. We show that there is some pair $\{v_i, v_{i+5}\}$ with each vertex having degree at least 3. This will suffice. By the degree sum of S, there are at least $t-1$ vertices in each of X and Y with degree at least 3. Thus by Lemma 1.1 we may infer that at least $t-1$ vertices of each of $V(C) \cap X$ and $V(C) \cap Y$ have degree at least 3. We consider now two cases.

When $t = 3$ there are at least two vertices in $V(C) \cap X$ that have degree at least 3. Every vertex in $V(C) \cap Y$ is distance 5 from one of these vertices, so since at least one of the vertices on $V(C) \cap Y$ has degree at least 3, we are done.

When $t \geq 4$ there are at least three vertices in $V(C) \cap X$ that have degree at least three. So there are at least three vertices in $V(C) \cap Y$ that distance 5 from one of these vertices. Since at most 2 vertices of $V(C) \cap Y$ have degree less than 3 at least one of these 3 vertices has degree at least 3. □

Now v_1 has neighbor y in $Y - V(C)$ and v_6 has neighbor x in $X - V(C)$. If $x \sim y$, then we have the C_{2t}

$$v_6, v_7, \ldots, v_{2t+2}, v_1, y, x, v_6.$$

On the other hand, if $x \nsim y$ then we have the alternating path $v_1 \sim y \nsim x \sim v_6 \nsim v_1$. Reversing this in G we arrive at a realization of the same degree sequence that contains a non-induced C_{2t+2}. By Claim 4.3, this suffices to complete the proof.

□

5. Conclusion

This paper serves only as an initial investigation into the subject of potentially H-bigraphic sequence pairs. Looking forward, it may be interesting to consider other broad classes of bipartite graphs, particularly those graphs H for which the standard potential number is known.

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