Linear Algebra - MATH 221 TEST Section 1.1 - 2.5

Name: SOLUTION KEY

Instructions: *The Emory Honor Code will be observed.* No calculators are allowed. Show all work to receive full credit. Be as specific and detailed as possible.

PLEASE EXPECT A MORE CHALLENGING AND LONGER EXAM THAN THIS ONE!!

1. Define a **one-to-one** mapping.

   A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b} \in \mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

2. **True or False:** Justify each answer.

   (a) If a system of linear equations has two different solutions, it must have infinitely many solutions.

   TRUE: If it has two different solutions then there must exist a free variable. We are thus guaranteed an infinite number of solutions.

   (b) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if there are no free variables.

   TRUE: see the discussion in section 1.5

   (c) Let A, B, C be matrices then $(AB)C = (AC)B$.

   FALSE: Matrix multiplication is not commutative in general. Provide an example to illustrate.

   (d) If $AC = 0$, then either $A = 0$ or $C = 0$.

   FALSE: Provide a counter-example.

   (e) If $AB = I_n$, then A is invertible.

   TRUE: Follows from the Invertible Matrix Theorem.

3. An indexed set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_p}\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if ...

   the vector equation

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} \ldots + x_p\mathbf{v_p} = \mathbf{0}$$

   has only the trivial solution.

4. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates points through an angle of $\pi/2$ radians about the origin. Find the standard matrix of T.

   To determine the standard matrix it is sufficient to know what the transformation does to the unit vectors, $\mathbf{e_1}, \mathbf{e_2}$.

   $A = \begin{bmatrix} T(\mathbf{e_1}) & T(\mathbf{e_2}) \end{bmatrix}$

   Thus,

   $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

5. Determine which of the following sets is linearly independent. Give reasons for your answers, use as few computations as possible.

   (a) The second vector is a scalar multiple of the first, thus this set is linearly dependent.

   $$v_1 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$
   $$v_2 = \begin{bmatrix} -40 \\ -80 \\ -120 \end{bmatrix}$$
   $$v_3 = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

   (b) This set contains the zero vector, so by Theorem 9 of chapter 1 we have that the set is linearly dependent.

   $$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
   $$v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
   $$v_3 = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

   (c) This set contains more vectors then there are entries in each vector, so by Theorem 8 of Chapter 1 we have that the set is linearly dependent.

   $$v_1 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$
   $$v_2 = \begin{bmatrix} -4 \\ -5 \\ -6 \end{bmatrix}$$
   $$v_3 = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$
   $$v_4 = \begin{bmatrix} 1 \\ 3 \\ 11 \end{bmatrix}$$

6. Given the LU factorization of the matrix A, solve the equation $A\mathbf{x} = \mathbf{b}$.

   Solution to be provided prior to next exam. LU **factorization not covered on first exam.**

   $L = \begin{bmatrix} 1 & 0 \\ -3/2 & 1 \end{bmatrix}$

   $U = \begin{bmatrix} 2 & 5 \\ 0 & 7/2 \end{bmatrix}$

   and

   $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

7. Explain the advantage, being as specific as possible, of a square matrix, A, having block upper triangular form. That is the matrix has the form

   Solution to be provided prior to next exam. Matrix partitions not covered on first exam.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

8. If an $n \times n$ matrix K cannot be row reduced to I_n, what can you say about the columns of K? Why?

   As K is not row equivalent to the identity matrix, the matrix K, by the Invertible Matrix Theorem, is not invertible, that is K is singular. Furthermore, the IMT implies that the columns of K *do not* form a linearly independent set and that the columns of K *do not* span $\mathbb{R}^n$.

9. Let

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$$

and $\mathbf{b} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$.

Find A^{-1} and use it to solve the equation $A\mathbf{x} = \mathbf{b}$.

According to Theorem 4 of Chapter 2, we find that

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}.$$

That is,

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}.$$

Thus, $\mathbf{x} = A^{-1}\mathbf{b}$ and

$$\mathbf{x} = \begin{bmatrix} 11 \\ -5 \end{bmatrix}$$

10. Do the columns of the following matrix span $\mathbb{R}^3$?

$$M = \begin{bmatrix} 4 & 0 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & -5 \end{bmatrix}$$

Yes. The matrix contains three pivots, so by the Invertible Matrix Theorem the columns of the matrix do span $\mathbb{R}^3$.

11. Describe your favorite linear system.

Answers will vary.