Given the following matrix:

\[ A = \begin{bmatrix} 10 & -8 \\ 2 & 2 \end{bmatrix} \]

(a) Determine the characteristic polynomial of \( A \).

(b) Determine the eigenvalues of \( A \).

(c) For each of the eigenvalues determine a basis for the corresponding eigenspace.
(d) Explain why, or why not, the matrix $A$ is diagonalizable (you need not give the diagonalization if one exists).

(2) A matrix $A$ is diagonalizable, i.e. $A = PDP^{-1}$ where $P$ and $D$ are given below. Use this diagonalization to compute $A^{10}$.

$$P = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$
(3) Without doing Gaussian elimination, write $y$ as a linear combination of the orthogonal basis $S$ for $\mathbb{R}^3$, where $S = \{u_1, u_2, u_3\}$.

\[
y = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}
\]

\[
u_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}
\]

\[
u_2 = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}
\]

\[
u_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}
\]
(4) With respect to the vectors of the previous problem, find the projection of $y$ onto $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and the orthogonal component of $y$. 
(5) Determine an orthogonal basis for the column space of $B$, where:

$$B = \begin{bmatrix}
  3 & 0 & 0 & 1 \\
  0 & 2 & -1 & 1 \\
  0 & 0 & 0 & 1 \\
  6 & 0 & 0 & 0
\end{bmatrix}$$

Now produce an orthonormal basis for $ColB$. 

(6) Find least-squares solutions of the equation $Ax = b$ by constructing the normal equations for $\hat{x}$ and solving for $\hat{x}$.

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

Compute the least-squares error associated with this solution.
(7) Prove that if \( S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\} \) is an orthogonal set of nonzero vectors in \( \mathbb{R}^n \), then \( S \) is linearly independent and hence is a basis for the subspace spanned by \( S \).
(8) John is either happy or sad. If he is happy one day, then he is happy the next day four times out of five. If he is sad one day, then he is sad the next day one time out of three. Over the long term, what are the chances that John is happy on any given day?

(You may present the solution to this problem in one of two ways: directly solve the problem or, if you’ve had enough of computation and don’t like fractions, explicitly and succinctly describe the steps you would take to solve it.)