LINEAR ALGEBRA
EXAM 3
SPRING 2026

Average
73/85

SD
5 points

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Signature: C. F. Gauss

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. The exam is proctored by permission of the Dean of the Faculty. Good luck!

(1) [5 points] Check whether the given vector is an eigenvector of the given matrix.

$$P = \begin{bmatrix} 5 & 0 & -1 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ 4 & -2 & -2 & 4 \end{bmatrix}, \ \mathbf{a} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 0 \end{bmatrix}$$

If it is an eigenvector, state the corresponding eigenvalue. If it is not an eigenvector, say why it isn't.

To easily check this, we multiply $\vec{a}$ by P to see if we obtain a scalar multiple of $\vec{a}$.

$P\vec{a} = \begin{bmatrix} 12 \\ 12 \\ 12 \\ 0 \end{bmatrix}$ and we see that we obtain $4\vec{a}$. Thus, $\vec{a}$ is an eigenvector w/ eigenvalue 4.

Date: May 14, 2026.

(2) [25 points] Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

(a) Find the eigenvalues of A. Let λ_1 denote the largest of these, λ_2 denote the next largest, etc.

To find the eigenvalues, we find the characteristic polynomial: $\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 4 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

So eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 1$

(b) Find a basis for each of the eigenvalues. State the dimension of each of the eigenspaces.

We find a basis for $\lambda_1 = 3$ by solving

$A\vec{x} = 3\vec{x}$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_2 \text{ is free} \\ x_1 = +x_2 \end{array}$$

$$\text{so } \vec{x} = \begin{bmatrix} +x_2 \\ +x_2 \end{bmatrix} = x_2 \begin{bmatrix} +1 \\ +1 \end{bmatrix}$$

A basis for $\lambda_1 = 3$ is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ w/ dimension 1.

We find a basis for $\lambda_2 = 1$ by solving $A\vec{x} = 1\vec{x}$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{so } \vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

A basis for $\lambda_1 = 3$ is $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ w/ dimension 1.

(c) Find a diagonalization of A.

A diagonalization for A exists by the Diagonalization Theorem: we have "enough" eigenvectors.

$A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

and $$P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

(d) Use this diagonalization to compute the 10^{th} power of A. (You may leave your answer in factored form.)

So, $A^{10} = (PDP^{-1})^{10}$

$$= PD^{10}P^{-1}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3^{10} & 0 \\ 0 & 1^{10} \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 3^{10} & -1 \\ 3^{10} & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3^{10}+1}{2} & \frac{3^{10}-1}{2} \\ \frac{3^{10}-1}{} & \frac{3^{10}+1}{} \end{bmatrix}$$

(e) Give the matrix for this transformation relative to the basis formed from the eigenvectors you obtained.

By the main theorem in Section 5.4,

this is $D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

(3) [10 points] Each of the statements is false. Demonstrate this by giving a counter-example

(a) (Slightly tricky) If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, then λ is an eigenvalue of A.

To be an eigenvector, $\vec{x}$ must be non-zero. Thus, the only possibility here for a counter-example is $\vec{x} = \vec{0}$. We can then choose λ to be a number that does not solve the characteristic equation. Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\lambda = 6$.

(b) For any scalar c and any $\mathbf{v} \in \mathbb{R}^3$, we have

$$\|c\mathbf{v}\| = c\|\mathbf{v}\|$$

Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $c = -1$ then

$\| c\vec{v} \| \neq -1 \| \vec{v} \|$

as the norm of any vector must be non-negative.

(c) For a set S of three vectors $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ in $\mathbb{R}^3$ for which $\mathbf{u_i} \cdot \mathbf{u_j} = 0$ where $1 \leq i < j \leq 3$, S is orthonormal.

Let $S = \left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$

then $\vec{u}_i \cdot \vec{u}_j = 0$ for any $i \neq j$ but the vectors do not have unit length, which is a requirement for orthonormal.

(4) [5 points] Suppose that $\mathbf{y} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$ and let W consist of all $\mathbf{x} \in \mathbb{R}^3$ such that $\mathbf{y} \cdot \mathbf{x} = 0$. Give a geometric description of W and prove that W is a subspace of $\mathbb{R}^3$.

W is a plane through the origin perpendicular to $\vec{y}$: it is the set of solutions to $5x_1 + 5x_2 + 5x_3 = 0$, which has one pivot and 2 free variables.

Showing that W is a subspace:

① $\vec{0} \in W$ since $\vec{0} \cdot \vec{y} = 0$.

② if $\vec{u}, \vec{v} \in W$ then $\vec{u} + \vec{v} \in W$ since $\vec{y} \cdot (\vec{u} + \vec{v}) = \vec{y} \cdot \vec{u} + \vec{y} \cdot \vec{v} = 0 + 0 = 0$.

③ if c is a scalar, $\vec{u} \in W$ then $c\vec{u} \in W$ since $\vec{y} \cdot c\vec{u} = c(\vec{y} \cdot \vec{u}) = c\,0 = 0$

Finding a basis for W would also be fruitful for establishing that W is a subspace.

(5) [5 points]. Compute the orthogonal projection of $\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ onto $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and compute the distance from $\mathbf{b}$ to the line spanned by $\mathbf{u}$.

$$\text{proj}_{\vec{u}} \vec{b} = \hat{b} = \frac{\vec{b} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{9}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

The distance from $\vec{b}$ to the line spanned by $\vec{u}$ is

$$\|\vec{b} - \hat{b}\| = \left\| \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} - \frac{9}{5}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 6/5 \\ -3/5 \\ 3 \end{bmatrix} \right\| = \sqrt{\frac{36}{25} + \frac{9}{25} + \frac{225}{25}}$$

$$= \sqrt{\frac{270}{25}} = \sqrt{\frac{54}{5}}$$

(7) [10 points] Let $A = [\mathbf{b_1}\ \mathbf{b_2}\ \mathbf{b_3}]$, where $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}$ are given in the previous problem. If $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, then $A\mathbf{x} = \mathbf{b}$ is inconsistent. Solve the general least-squares problem AND give the least-squares error. Use any method you wish.

We set up and solve the normal equations: $A^TA\vec{x} = A^T\vec{b}$;

where $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

Then $A^TA = \begin{bmatrix} 6 & 3 & 9 \\ 3 & 3 & 5 \\ 9 & 5 & 14 \end{bmatrix}$ and $A^T\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

We perform G.E. on $\left[\begin{array}{ccc|c} 6 & 3 & 9 & 1 \\ 3 & 3 & 5 & 0 \\ 9 & 5 & 14 & 1 \end{array}\right]$ ..... a lot of computation here....

Solving this we get $\hat{x} = \begin{bmatrix} 5/3 \\ 0 \\ -1 \end{bmatrix}$.

$$A\hat{x} = \frac{5}{3}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 1/3 \\ 0 \\ 2/3 \end{bmatrix}$$

The least squares error is

$$\|\vec{b} - A\hat{x}\| = \left\| \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/3 \\ 1/3 \\ 0 \\ 2/3 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 1/3 \\ -1/3 \\ 0 \\ 1/3 \end{bmatrix} \right\| = \sqrt{1/9 + 1/9 + 1/9} = \frac{\sqrt{3}}{3}.$$

Alternately, one could use the orthogonal basis found in the previous problem! Compute $\hat{b}$ using it and $\hat{x}$ is simultaneously obtained (see Example 4 on page 364).

(6) [10 points] Apply the Gram-Schmidt Process to the following set of vectors to form an orthogonal basis for the space spanned by these vectors.

$$\{\mathbf{b_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b_3} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}\}$$

We let $\vec{v}_1 = \vec{b}_1$.

We set $\vec{v}_2 = \vec{b}_2 - \text{proj}_{\vec{v}_1} \vec{b}_2$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1 \\ -1/2 \end{bmatrix}$$

Then $\vec{v}_3 = \vec{b}_3 - \text{proj}_{\text{Span}\{\vec{v}_1, \vec{v}_2\}} \vec{b}_3$

$$= \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} - \frac{9}{6} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1/2 \\ 0 \\ 1 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 0 \\ -1/3 \\ -1/3 \end{bmatrix}$$

Thus, an orthogonal basis is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

To be "extra sure", one could check orthogonality by computing $\vec{v}_1 \cdot \vec{v}_2$, $\vec{v}_1 \cdot \vec{v}_3$, and $\vec{v}_2 \cdot \vec{v}_3$.

(8) [15 points] **Google's PageRank Algorithm**. The article by K. Bryan and T. Leise gives us the formula

$$\mathbf{M} = (1 - m)\mathbf{A} + m\mathbf{S},$$

where $0 \leq m \leq 1$, $\mathbf{S}$ is the $n \times n$ matrix with all entries $1/n$, and A is the link matrix. Give A for the network drawn below. What structure does A have? What does this imply about the solutions to $A\mathbf{x} = \mathbf{x}$?[1] Then give $\mathbf{M}$ where $m = 1/10$.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

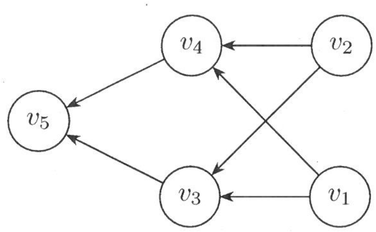
Two things to note about A:
① it is not column stochastic due to column 5
② it is lower-triangular

Both of these guarantee that solving $A\vec{x} = \vec{x}$ will give multiple free variables, so the dimension of the eigen space is greater than 1.

Then

$$M = \frac{9}{10} A + \frac{1}{10} S$$

where $S = \begin{bmatrix} 1/5 & \cdots & 1/5 \\ \vdots & & \vdots \\ 1/5 & & 1/5 \end{bmatrix}$.



[1] I would hope that you could answer this question without doing any Gaussian Elimination but just thinking what the result would have to be at the end of G.E. It is also best to write A with column and row labels that honors the vertex labeling given.

Fill in the following blanks.

To determine a ranking for the webpages, we use **M** to find the importance scores by solving the equation $M\vec{x} = \vec{x}$.

That is, we find an eigenvector corresponding to the eigenvalue 1.

Furthermore, each entry of the matrix M is strictly positive.

This is a key property for it guarantees that the eigenspace corresponding to this eigenvalue is one-dimensional.