LINEAR ALGEBRA
EXAM 3
SPRING 2026

Name:
Honor Code Statement:

Signature:

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. The exam is proctored by permission of the Dean of the Faculty. Good luck!

(1) [5 points] Check whether the given vector is an eigenvector of the given matrix.

$$P= \begin{bmatrix} 5 & 0 & -1 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ 4 & -2 & -2 & 4 \end{bmatrix}, \mathbf{a} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 0 \end{bmatrix}$$

If it is an eigenvector, state the corresponding eigenvalue. If it is not an eigenvector, say why it isn't.

Date: May 14, 2026.

(2) [25 points] Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

(a) Find the eigenvalues of A. Let λ_1 denote the largest of these, λ_2 denote the next largest, etc.

(b) Find a basis for each of the eigenvalues. State the dimension of each of the eigenspaces.

(c) Find a diagonalization of A.

(d) Use this diagonalization to compute the 10^{th} power of A. (You may leave your answer in factored form.)

(e) Give the matrix for this transformation relative to the basis formed from the eigenvectors you obtained.

(3) [10 points] Each of the statements is false. Demonstrate this by giving a counter-example

(a) (Slightly tricky) If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, then λ is an eigenvalue of A.

(b) For any scalar c and any $\mathbf{v} \in \mathbb{R}^3$, we have

$$\|c\mathbf{v}\| = c\|\mathbf{v}\|$$

.

(c) For a set S of three vectors $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ in $\mathbb{R}^3$ for which $\mathbf{u_i} \cdot \mathbf{u_j} = 0$ where $1 \leq i < j \leq 3$, S is orthonormal.

(4) [5 points] Suppose that $\mathbf{y} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$ and let W consist of all $\mathbf{x} \in \mathbb{R}^3$ such that $\mathbf{y} \cdot \mathbf{x} = 0$. Give a geometric description of W and prove that W is a subspace of $\mathbb{R}^3$.

(5) [5 points]. Compute the orthogonal projection of $\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ onto $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and compute the distance from $\mathbf{b}$ to the line spanned by $\mathbf{u}$.

(6) [10 points] Apply the Gram-Schmidt Process to the following set of vectors to form an orthogonal basis for the space spanned by these vectors.

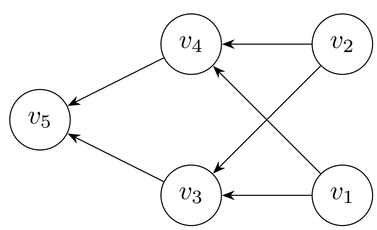
$$\{\mathbf{b_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b_3} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}\}$$

(7) [10 points] Let $A = [\mathbf{b_1}\ \mathbf{b_2}\ \mathbf{b_3}]$, where $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}$ are given in the previous problem. If $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, then $A\mathbf{x} = \mathbf{b}$ is inconsistent. Solve the general least-squares problem AND give the least-squares error. Use any method you wish.

(8) [15 points] **Google's PageRank Algorithm**. The article by K. Bryan and T. Leise gives us the formula

$$\mathbf{M} = (1 - m)\mathbf{A} + m\mathbf{S},$$

where $0 \leq m \leq 1$, $\mathbf{S}$ is the $n \times n$ matrix with all entries $1/n$, and A is the link matrix. Give A for the network drawn below. What structure does A have? What does this imply about the solutions to $A\mathbf{x} = \mathbf{x}$?[1] Then give $\mathbf{M}$ where $m = 1/10$.



[1] I would hope that you could answer this question without doing any Gaussian Elimination but just thinking what the result would have to be at the end of G.E. It is also best to write A with column and row labels that honors the vertex labeling given.

Fill in the following blanks.

To determine a ranking for the webpages, we use **M** to find the importance scores by solving the equation ____. That is, we find an eigenvector corresponding to the eigenvalue ________________. Furthermore, each entry of the matrix M is strictly ________________________________. This is a key property for it guarantees that the eigenspace corresponding to this eigenvalue is ________________________________.