75 points total.

LINEAR ALGEBRA
EXAM 3
SPRING 2017

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.
Signature: C.F. Gauss.

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Please turn off all electronic devices – in fact, you shouldn't have any with you. Additional blank white paper is available at the front of the room – you are not permitted to use any other paper. Good luck!

(1) [10 points] Apply the Gram-Schmidt Process to the following set of vectors in $\mathbb{R}^3$, $\mathbf{u_1} = (1,1,1), \mathbf{u_2} = (0,1,1), \mathbf{u_3} = (0,0,1)$. Then normalize the vectors you obtain.

Let $\vec{v}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Let $\vec{v}_2 = \vec{u}_2 - \dfrac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$

Let $\vec{v}_3 = \vec{u}_3 - \dfrac{\vec{u}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \dfrac{\vec{u}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \dfrac{1/3}{6/9} \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} - \begin{bmatrix} -1/3 \\ 1/6 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \end{bmatrix}$$

Thus, by the Gram-Schmidt Process, we have that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \end{bmatrix} \right\}$ is an orthogonal set w/ the same span as the original set.

We now normalize the obtained vectors. To do so, we compute the norm (length) of each and multiply them by the reciprocal of it.

$\|\vec{v}_1\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$; $\|\vec{v}_2\| = \sqrt{4/9+1/9+1/9} = \sqrt{2/3}$; $\|\vec{v}_3\| = \sqrt{0+1/4+1/4} = \sqrt{1/2}$

We obtain $\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -2\sqrt{6}/6 \\ 1\sqrt{6}/6 \\ \sqrt{6}/6 \end{bmatrix}, \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \right\}$

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(2) [5 points] Find the eigenvalues of the following matrix A. [5 points] Then find an basis for each eigenspace. [2 points] Use the eigenvalues to determine whether or not the matrix is invertible. [3 points] State why or why not the matrix is diagonalizable.

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

We wish to find the λ that satisfy $A\vec{x} = \lambda x$

This is equivalent to knowing when $\det(A - \lambda I) = 0$.

So,

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{bmatrix} = A - \lambda I$$

$$\det(A - \lambda I) = (1-\lambda)(2-\lambda) - 12 = 2 - 3\lambda + \lambda^2 - 12$$
$$= -10 - 3\lambda + \lambda^2 = (-5 + \lambda)(2 + \lambda)$$

We set this to zero to find $\lambda = 5$ or $\lambda = -2$ are eigenvalues

We now find an basis for each eigenspace:

For $\lambda = 5$: $A\vec{x} = 5\vec{x} \Rightarrow \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix}\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} -4 & 3 & 0 \\ 4 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -4 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$.

So x_2 is free and $4x_1 = 3x_2$, $x_1 = \frac{3}{4}x_2$ Thus $\vec{x} = \begin{bmatrix} 3/4\, x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$.

So $\left\{ \begin{bmatrix} 3/4 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace corresponding to $\lambda = 5$.

For $\lambda = -2$ $A\vec{x} = -2\vec{x} \Rightarrow (A + 2I)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 4 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

So x_2 is free and $-x_1 = x_2$. Thus $\vec{x} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

So $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace corresponding to $\lambda = -2$.

Also, as zero is not an eigenvalue, the matrix is invertible according to the IMT.

Lastly, ~~As~~ as A is 2×2 and has 2 distinct eigenvalues it is diagonalizable.

(3) [15 points - 5 points for each part] Find (1) the orthogonal projection of $\mathbf{b}$ onto Col A, (2) a least-squares solution of $A\mathbf{x} = \mathbf{b}$, and (3) the least-squares error.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

and $\mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$

(1) One may check that the columns of A form an orthogonal set. Thus, we may use the Orthogonal Decomposition Theorem to find $\hat{b} = \text{proj}_{\text{Col}A} \vec{b}$.

Thus,

$$\hat{b} = \frac{\vec{b} \cdot \vec{a}_1}{\vec{a}_1 \cdot \vec{a}_1} \vec{a}_1 + \frac{\vec{b} \cdot \vec{a}_2}{\vec{a}_2 \cdot \vec{a}_2} \vec{a}_2 + \frac{\vec{b} \cdot \vec{a}_3}{\vec{a}_3 \cdot \vec{a}_3} \vec{a}_3$$

$$\hat{b} = \frac{1}{3} \vec{a}_1 + \frac{14}{3} \vec{a}_2 - \frac{5}{3} \vec{a}_3 = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \frac{14}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{3} \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

(2) The least-squares solution is the $\hat{x}$ such that $A\hat{x} = \hat{b}$. Given this and the above, we see that $\hat{x} = \begin{bmatrix} 1/3 \\ 14/3 \\ -5/3 \end{bmatrix}$, which is found by "reading" the weights in the above expression of $\hat{b}$.

(3) The least-squares error is given by

$$\| \vec{b} - A\hat{x} \| = \| \vec{b} - \hat{b} \| = \left\| \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -3 \\ 3 \\ 3 \\ 0 \end{bmatrix} \right\|$$

$$= \sqrt{(-3)^2 + 3^2 + 3^2 + 0^2} = \sqrt{27} = 3\sqrt{3}.$$

(4) [3 points each] Each of the following statements is false. Amend each statement in as few words or symbols as possible, without insertion or deletion of the word "not" (that is, without negating the conclusion), to make a true statement or explain why it's false, which can be done by pointing to a theorem or giving a counter-example.

(a) The eigenvalues of a 2×2 matrix are on its main diagonal.

write "2x2 triangular matrix" to make a true statement

(b) An $n \times n$ matrix A is diagonalizable if A has n eigenvectors.

write "if A has n linearly independent eigenvectors"

(c) For any $\mathbf{v} \in \mathbb{R}^3$ any scalar $c \in \mathbb{R}$, $\|c\mathbf{v}\| = c\|\mathbf{v}\|$.

A counter example: let $c = -1$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Then $\|c\vec{v}\| = \left\| \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\| = \sqrt{(-1)^2 + 0 + 0} = 1$

but $-1 \left\| \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\| = -1 \cdot \sqrt{1^2 + 0^2 + 0^2} = -1$

(d) If a set $S = \{\mathbf{u_1}, \ldots, \mathbf{u_p}\}$ has the property that $\mathbf{u_i} \cdot \mathbf{u_j} = 0$ whenever $i \neq j$, then S is an orthonormal set.

Either change orthonormal to orthogonal

or

insert that $\|\vec{u}_i\| = 1$ for $1 \leq i \leq p$.

(e) A row replacement operation on a matrix A does not change the eigenvalues.

A counter-example:

Note A from problem 2, which we saw has eigenvalues 5 and -2.

A is row equivalent to $\begin{bmatrix} 1 & 3 \\ 0 & -10 \end{bmatrix}$, which has eigenvalues 1 and -10 since we see it is a triangular matrix

(5) [5points] REMOVED FROM EXAM

(6) [10points] Define $T : \mathbb{P}_2 \to \mathbb{R}^3$ by $T(\mathbf{p}) = (\mathbf{p}(-1), \mathbf{p}(0), \mathbf{p}(1))$. Find the matrix for T relative to the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$ and the standard basis for $\mathbb{R}^3$.

We follow the procedure out-lined on page 289 of the text.

$T(\vec{b}_1) = T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $T(\vec{b}_2) = T(t) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$; $T(\vec{b}_3) = T(t^2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

$\left[T(\vec{b}_1)\right]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $\left[T(\vec{b}_2)\right]_{\mathcal{E}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$; $\left[T(\vec{b}_3)\right]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Thus the matrix for T relative to the given bases is

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

(7) [10 points] Prove the following theorem.

THEOREM If $S = \{\mathbf{u_1}, \dots, \mathbf{u_p}\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent.

This is Theorem 4 of Chapter 6.

To prove that S is linearly independent we need to show that the only solution to the following equation is the trivial one,

$$c_1\vec{u}_1 + \dots + c_p\vec{u}_p = \vec{0}. \qquad (1)$$

That is, we must show that in equation (1), we must have $c_1 = \dots = c_p = 0$.

Consider "dotting" both sides of equation (1) by $\vec{u_i}$.
Then by the orthogonality of the set and properties of the dot product, we have

$$c_i \vec{u_i} \cdot \vec{u_i} = 0$$

As S contains no zero vectors, $\vec{u_i} \cdot \vec{u_i} \neq 0$. Thus, c_i must equal 0, and this is true for $1 \leq i \leq p$. This completes the proof.