LINEAR ALGEBRA
EXAM 3
SPRING 2014

Name: Solution Key.

Honor Code Statement: I have neither given nor received any unauthorized aid on this exam.

Signature:

Total 70 points.

Avg: 52/70

SD: 5.55

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Please turn off all electronic devices – in fact, you shouldn't have any with you. Additional blank white paper is available at the front of the room – you are not permitted to use any other paper. Good luck!

(1) [5 points] Let T_1 be a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that reflects points across the line $y = \frac{1}{2}x$. Let T_2 be a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that reflects points across the line $y = -x$. If A is the associated matrix for $T_1(T_2(\mathbf{x}))$, find all the real eigenvalues of A and give one real eigenvector for each eigenvalue, each with unit length. (Note: you needn't determine A to do this, though it's OK if you want. And, if it helps, there is square paper at the front of the room.)

Solution 1

The standard matrix for the transformation T_2 is $A_2 = \begin{bmatrix} T_2(\vec{e}_1) & T_2(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

The standard matrix for the transformation T_1 is $A_1 = \begin{bmatrix} T_1(\vec{e}_1) & T_1(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

Thus the matrix for the transformation $T_1 \circ T_2$ is $A_1A_2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$= A.$

Find the eigenvalues of A: $\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1/\sqrt{2} \\ 1/\sqrt{2} & (-1/\sqrt{2} - \lambda) \end{vmatrix} = 0 \Rightarrow \lambda^2 + \lambda/\sqrt{2} + 1/2 = 0$

By the quadratic formula, $\lambda = \dfrac{-1/\sqrt{2} \pm \sqrt{-3/2}}{2}$. Thus, no real eigenvalues. Thus, no real eigenvectors.

(Given my omission of the word "real" in the problem statement, I was "generous" in grading.)

Date: May 15, 2014.

Solution 2. Note that reflection over a line preserves the length of any vector. Thus, the only possible real eigenvalues are ± 1. This would require any associated eigenvectors to be on the "folding" lines, i.e. the lines of reflection. Do a little folding and see that this isn't possible.

(2) [5 points] The following set of vectors $\{\mathbf{b_1}=\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{b_2}=\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{b_3}=\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}\}$ is an orthogonal basis for $\mathbb{R}^3$.

Let $\mathbf{x}=\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$.

Find the projection of $\mathbf{x}$ on the following vector spaces: Span$\{\mathbf{b_1}\}$, Span$\{\mathbf{b_2}\}$ and Span$\{\mathbf{b_3}\}$. Then use this information to express $\mathbf{x}$ as a linear combination of the vectors $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}$.

$$\text{proj}_{\text{Span}\{\vec{b}_1\}} \vec{x} = \frac{\vec{x}\cdot\vec{b}_1}{\vec{b}_1\cdot\vec{b}_1}\vec{b}_1 = \frac{0}{18}\vec{b}_1 = \vec{0}$$

$$\text{proj}_{\text{Span}\{\vec{b}_2\}} \vec{x} = \frac{\vec{x}\cdot\vec{b}_2}{\vec{b}_2\cdot\vec{b}_2}\vec{b}_2 = \frac{15}{9}\vec{b}_2 = \frac{5}{3}\vec{b}_2 = \begin{bmatrix} 10/3 \\ 10/3 \\ -5/3 \end{bmatrix}$$

$$\text{proj}_{\text{Span}\{\vec{b}_3\}} \vec{x} = \frac{\vec{x}\cdot\vec{b}_3}{\vec{b}_3\cdot\vec{b}_3}\vec{b}_3 = \frac{30}{18}\vec{b}_3 = \frac{5}{3}\vec{b}_3 = \begin{bmatrix} 5/3 \\ 5/3 \\ 20/3 \end{bmatrix}$$

$$\Rightarrow \quad \vec{x} = 0\vec{b}_1 + \frac{5}{3}\vec{b}_2 + \frac{5}{3}\vec{b}_3$$

(3) [5 points] With respect to the basis from the previous problem, one could perform Gaussian elimination on the matrix $[\mathbf{b_1}\ \mathbf{b_2}\ \mathbf{b_3}]$ and find 3 pivot columns to show that the set is linearly independent. How might you use the dot product to establish this fact?

Using the dot product, ~~it~~ we are told ~~establish~~ the set is an orthogonal set, then by Theorem 4 of Section 6.2, we know it is a linearly independent set.

Note
$$\vec{b}_1\cdot\vec{b}_2 = 6 - 6 + 0 = 0$$
$$\vec{b}_1\cdot\vec{b}_2 = 3 - 3 + 0 = 0$$
$$\vec{b}_2\cdot\vec{b}_3 = 2 + 2 - 4 = 0.$$

Thus, the set is orthogonal and thus linearly independent. since the vector equation $\vec{0} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3$ has only the trivial solution. Indeed, consider $\vec{b}_1\cdot\vec{0} = \vec{b}_1\cdot(c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3) \Rightarrow 0 = c_1\vec{b}_1\cdot\vec{b}_1$ $\Rightarrow c_1 = 0$. This way we can also show $c_2, c_3 = 0$.

(4) [2 points each] Short answer (with *brief* justification):

(a) For a given 4×4 matrix A, the eigenvalues are 3 and 2, each with multiplicity two. Is A invertible?

As 0 is not an eigenvalue, by the Invertible Matrix Theorem, A is invertible.

(b) For a given 4×4 matrix A, the eigenvalues are 3 and 2, each with multiplicity two. Is A diagonalizable?

We don't know from this information whether or not A is invertible. We'd need to know about the dimension of each eigenspace.

(c) For a vector space W, give a vector that is in W and $W^\perp$.

Every vector space contains the zero vector, and the dot product of the zero vector and any $w \in W$ is 0; thus the zero vector is also in $W^\perp$. So the zero vector is in both.

(d) **True or False:** The Orthogonal Decomposition Theorem is used to establish the Best Approximation Theorem, but not the Gram-Schmidt Process.

False. It is used to establish each of these other results.

(e) **True or False:** If $\mathbf{x}$ is an eigenvector of A, then it is also an eigenvector of A^2.

True. If $\vec{x}$ is an eigenvector of A, then $A\vec{x} = \lambda\vec{x}$ for some real number λ and non zero vector $\vec{x}$.

Multiplying both sides by A, we obtain

$$A^2\vec{x} = A(\lambda\vec{x}) = \lambda A\vec{x} = \lambda(\lambda\vec{x}) = \lambda^2\vec{x}.$$

Thus $A^2\vec{x} = \lambda^2\vec{x}$, i.e. multiplication of $\vec{x}$ by A is the same as scalar multiplication of $\vec{x}$ by λ^2.

(5) [5 points] Prove that the dot product is commutative for vectors in $\mathbb{R}^3$.

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$. We must show $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

By definition $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$,

and $\vec{v} \cdot \vec{u} = v_1 u_1 + v_2 u_2 + v_3 u_3$.

As ~~mult~~ multiplication of real numbers is commutative, we may write $u_i v_i$ in place of $v_i u_i$ for $1 \le i \le 3$. The result follows.

(6) [5 points] Define **eigenspace**.

The eigenspace of a matrix A corresponding to a real number λ is the set of all solutions of $(A - \lambda I)\vec{x} = \vec{0}$, and where λ is an eigenvalue.

and is $B' = \{1, t, t^2, t^3, t^4\}$ for $\mathbb{P}_4$.

(7) [5 points] Let T be a linear transformation from $\mathbb{P}_3$ to $\mathbb{P}_4$. The basis for $\mathbb{P}_3$ is $\mathbf{B} = \{1, t, t^2, t^3\}$. If T is the integration operator (from Calculus I) with constant of integration zero, then find the matrix for T relative to $\mathbf{B}$ and B'.

The defined mapping is $T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = 0 + a_0 t + \frac{a_1}{2} t^2 + \frac{a_2}{3} t^3 + \frac{a_3}{4} t^4$, which maps from $\mathbb{P}_3$ to $\mathbb{P}_4$.

We will first compute the images of the basis vectors.

$$\left.\begin{aligned} T(1) &= t \\ T(t) &= \frac{t^2}{2} \\ T(t^2) &= \frac{t^3}{3} \\ T(t^3) &= \frac{t^4}{4} \end{aligned}\right\}$$

We then write the B-coordinate vectors of these, where we extend B to $\{1, t, t^2, t^3, t^4\} = B'$ as the basis for $\mathbb{P}_4$.

$$[T(1)]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [T(t)]_B = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}$$

$$[T(t^2)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/3 \\ 0 \end{bmatrix} \text{ and } [T(t^3)] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/4 \end{bmatrix}.$$

Thus, the matrix for T relative to these bases is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}_{5\times 4}$$

(8) [10 points] Given the following matrix A, make the following computations.

$$A=\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

(a) Find the two distinct eigenvalues of A.

As A is triangular, Theorem 1 of Section 5.1 tells us that the eigenvalues are the entries on the main diagonal: 2 and 1.

(b) Find a basis for each of the corresponding eigenvalues.

First, we solve $A\vec{x} = 2\vec{x}$, i.e. $A\vec{x} - 2\vec{x} = \vec{0}$.

So we consider ~~perfor~~ the following matrix

$\left[\begin{array}{cc|c} -1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right]$ and perform G.E. to obtain

$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$. Thus $x_1 = 0$ and x_2 is free. So $\vec{x} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Thus, a basis for the eigenspace corresponding to 2 is $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

We now do the same for $\lambda = 1$.

$A\vec{x} - \vec{x} = \vec{0}$. $\Rightarrow \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow x_1 = x_2$, x_2 is free $\Rightarrow \vec{x} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Thus, a basis for the eigenspace corresponding to 1 is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(c) Diagonalize A.

This is possible since the dimension of each eigenspace matches the multiplicity of the corresponding eigenvalue.

$A = PDP^{-1}$, where $P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$; as told to us by the Diagonalization Theorem.

Note that by Theorem 4 of Section 2.2, $P^{-1} = -1 \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

(d) Use this diagonalization to compute A^4.

Now $A^4 = (PDP^{-1})^4$

$= PD^4P^{-1}$

$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 16 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -15 & 16 \end{bmatrix}$

(9) [10 points] Describe all least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$, where A and $\mathbf{b}$ are as given below.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix}$$

We use Theorem 13 of Section 6.5 and begin by forming the normal equations:

$$A^T A \vec{x} = A^T \vec{b}$$

We see $A^T A = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix}$

and $A^T \vec{b} = \begin{bmatrix} 27 \\ 12 \\ 15 \end{bmatrix}$

Thus, we perform G.E. on $\left[\begin{array}{ccc|c} 6 & 3 & 3 & 27 \\ 3 & 3 & 0 & 12 \\ 3 & 0 & 3 & 15 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 4\,1/2 \\ 1 & 1 & 0 & 4 \\ 1 & 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 4\,1/2 \\ 0 & 1/2 & -1/2 & -1/2 \\ 0 & -1/2 & 1/2 & 1/2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 4\,1/2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = -x_3 + 5 \\ x_2 = +x_3 - 1 \\ x_3 \text{ is free.} \end{array}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} -x_3 + 5 \\ x_3 - 1 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}.$$

Solutions have this form.

(10) [10 points] **The 25 billion dollar eigenvector** Let A be the link matrix of a 5-node web in which page 5 has no backlinks. If we use an eigenvector of A to assign an importance score to page 5, what is the importance score? (Justify your answer.)

As page 5 has no backlinks, row 5 of matrix A is comprised entirely of zeros. Thus, multiply any $\vec{x} \in \mathbb{R}^5$ by A results in entry 5 being 0. Thus any eigenvector must have 0 as entry 5. Thus, the importance score of page 5 is 0.

OR From a philosophical point of view: as "importance" is derived from the importance of others and as no other webpage has transferred its importance to page 5, page 5 has no importance.

Now consider the same A as above, but use $M = (1-m)A + mS$, where S is the 5×5 matrix with all entries $1/5$. Let $m = 0.2$. If we use an eigenvector of M to assign an importance score to page 5, what is the importance score? (Justify your answer.)

First recall that the sum of the importance scores $x_1 + x_2 + x_3 + x_4 + x_5$ is 1. Next note that as the 5^{th} row of A is all zeros, the 5^{th} row of M has all entries equal to $m \cdot \frac{1}{5} = \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{25}$.

Now consider $M\vec{x}$, where $\vec{x}$ is eigenvector of importance scores, and focus on the row-column rule of multiplication w/ row 5.

$$M\vec{x} = \underbrace{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1/25 & 1/25 & \ldots & & 1/25 \end{bmatrix}}_{5\times 5} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

We get $\frac{1}{25}x_1 + \frac{1}{25}x_2 + \frac{1}{25}x_3 + \frac{1}{25}x_4 + \frac{1}{25}x_5$

$= \frac{1}{25}(x_1 + \ldots + x_5) = \frac{1}{25} \cdot 1 = \frac{1}{25}$.

And, this must ~~eq~~ also equal x_5, so page 5 has importance score of $1/25$.