LINEAR ALGEBRA
EXAM 3
SPRING 2014

Name:
Honor Code Statement:

Signature:

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Please turn off all electronic devices – in fact, you shouldn't have any with you. Additional blank white paper is available at the front of the room – you are not permitted to use any other paper. Good luck!

(1) [5 points] Let T_1 be a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that reflects points across the line $y = \frac{1}{2}x$. Let T_2 be a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that reflects points across the line $y = -x$. If A is the associated matrix for $T_1(T_2(\mathbf{x}))$, find all the real eigenvalues of A and give one eigenvector for each real eigenvalue, each with unit length. (Note: you needn't determine A to do this, though it's OK if you want. And, if it helps, there is square paper at the front of the room.)

Date: May 15, 2014.

(2) [5 points] The following set of vectors $\{\mathbf{b_1}=\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{b_2}=\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{b_3}=\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}\}$ is an orthogonal basis for $\mathbb{R}^3$.

Let $\mathbf{x}=\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$.

Find the projection of $\mathbf{x}$ on the following vector spaces: Span$\{\mathbf{b_1}\}$, Span$\{\mathbf{b_2}\}$ and Span$\{\mathbf{b_3}\}$. Then use this information to express $\mathbf{x}$ as a linear combination of the vectors $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}$.

(3) [5 points] With respect to the basis from the previous problem, one could perform Gaussian elimination on the matrix $[\mathbf{b_1}\ \mathbf{b_2}\ \mathbf{b_3}]$ and find 3 pivot columns to show that the set is linearly independent. How might you use the dot product to establish this fact?

(4) [2 points each] Short answer (with *brief* justification):

(a) For a given 4×4 matrix A, the eigenvalues are 3 and 2, each with multiplicity two. Is A invertible?

(b) For a given 4×4 matrix A, the eigenvalues are 3 and 2, each with multiplicity two. Is A diagonalizable?

(c) For a vector space W, give a vector that is in W and $W^{\perp}$.

(d) **True or False:** The Orthogonal Decomposition Theorem is used to establish the Best Approximation Theorem, but not the Gram-Schmidt Process.

(e) **True or False:** If $\mathbf{x}$ is an eigenvector of A, then it is also an eigenvector of A^2.

(5) [5 points] Prove that the dot product is commutative for vectors in $\mathbb{R}^3$.

(6) [5 points] Define **eigenspace**.

(7) [5 points] Let T be a linear transformation from $\mathbb{P}_3$ to $\mathbb{P}_4$. The basis for $\mathbb{P}_3$ is $\mathbf{B} = \{1, t, t^2, t^3\}$ and for $\mathbb{P}_4$ is **B'**$= \{1, t, t^2, t^3, t^4\}$. If T is the integration operator (from Calculus I) with constant of integration zero, then find the matrix for T relative to $\mathbf{B}$ and **B'**.

(8) [10 points] Given the following matrix A, make the following computations.

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

(a) Find the two distinct eigenvalues of A.

(b) Find a basis for each of the corresponding eigenvalues.

(c) Diagonalize A.

(d) Use this diagonalization to compute A^4.

(9) [10 points] Describe all least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$, where A and $\mathbf{b}$ are as given below.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix}$$

(10) [10 points] **The 25 billion dollar eigenvector** Let A be the link matrix of a 5-node web in which page 5 has no backlinks. If we use an eigenvector of A to assign an importance score to page 5, what is the importance score? (Justify your answer.)

Now consider the same A as above, but use $M = (1-m)A + mS$, where S is the 5×5 matrix with all entries $1/5$. Let $m = 0.2$. If we use an eigenvector of M to assign an importance score to page 5, what is the importance score? (Justify your answer.)