LINEAR ALGEBRA
EXAM 3
SPRING 2013

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.
Additional Honor Code Statement: I have not witnessed another giving or receiving unauthorized aid.
Signature: C. F. Gauss

Directions: Complete all problems. Justify all answers/solutions. Calculators, texts, and notes are not permitted. Please turn off all electronic devices. Additional blank white paper is available at the front of the room. Good luck!

(1) [5 points] Let T be a transformation in $\mathbb{R}^3$ that rotates points about some line through the origin. If A is the associated matrix, find an eigenvalue of A and describe the eigenspace that corresponds to this value.

As the transformation rotates points about some line, the vectors that "lie on this line" are fixed. That is, any $\vec{x}$ on this line satisfies $A\vec{x} = \vec{x}$. So we see that 1 is an eigenvalue of A and the corresponding eigenspace is the set of all vectors that lie on the line of rotation.

(2) [5 points] Define **orthonormal set**.

A set $\{\vec{u}_1, \ldots, \vec{u}_p\}$ is an orthonormal set if $\vec{u}_i \cdot \vec{u}_j = 0$ for $i \neq j$ and $\|\vec{u}_i\| = 1$ for $1 \leq i \leq p$.

Date: May 16, 2013.

(3) [20 points] Given the following matrix A, make the following computations.

$A=\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

(a) Find the two distinct eigenvalues of A.

We must find all scalars λ s.t. $(A - \lambda I)\vec{x} = \vec{0}$.

To do so, we solve $\det(A-\lambda I)=0$ for all possible λ (as a result of IMT)

$$\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda) - 12 = 2 - 3\lambda + \lambda^2 - 12$$

So we solve $\lambda^2 - 3\lambda - 10 = 0 \Rightarrow (\lambda-5)(\lambda+2) = 0$

$\lambda = 5$ and $\lambda = -2$ are the two eigenvalues.

(b) Find a basis for each of the corresponding eigenvalues.

To find a basis for $\lambda=5$, we solve the system $(A-5I)\vec{x}=\vec{0}$. We use Gaussian Elim.

$$\left[\begin{array}{cc|c} -4 & 3 & 0 \\ 4 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -4 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3/4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So x_2 is free, $x_1 = 3/4\, x_2$. Thus a basis for the eigenspace is $\left\{\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}\right\}$

To find a basis for $\lambda=-2$, we solve the system $(A+2I)\vec{x}=\vec{0}$.

We use Gaussian Elimination.

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 4 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So x_2 is free, $x_1 = -x_2$. Thus a basis for the eigenspace is $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

(c) Diagonalize A.

To diagonalize A we use the Diagonalization Theorem. The matrix A can be diagonalized since we have found $n=2$ linearly independent eigenvectors.

Thus, $P = \begin{bmatrix} 3/4 & -1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$. We can compute P^{-1}, it is $P^{-1} = \frac{1}{7/4}\begin{bmatrix} 1 & 1 \\ -1 & 3/4 \end{bmatrix}$

$$= \begin{bmatrix} 4/7 & 4/7 \\ -4/7 & 3/7 \end{bmatrix}$$

Thus, $A = PDP^{-1}$.

(d) Use this diagonalization to compute A^4.

$$A^4 = (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1})$$

$$= PD^4P^{-1} \quad \text{by associativity.}$$

So, $D^4 = \begin{bmatrix} 625 & 0 \\ 0 & 16 \end{bmatrix}$

$$A^4 = \begin{bmatrix} 3/4 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 625 & 0 \\ 0 & 16 \end{bmatrix}\begin{bmatrix} 4/7 & 4/7 \\ -4/7 & 3/7 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1875}{4} & -16 \\ 625 & 16 \end{bmatrix}\begin{bmatrix} 4/7 & 4/7 \\ -4/7 & 3/7 \end{bmatrix} = \begin{bmatrix} \frac{1875+64}{7} & \frac{1875-48}{7} \\ \frac{2500-64}{7} & \frac{2500+48}{7} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1939}{7} & \frac{1827}{7} \\ \frac{2436}{7} & \frac{2548}{7} \end{bmatrix} = \begin{bmatrix} 277 & 261 \\ 348 & 364 \end{bmatrix}$$

(sorry for the arithmetic!)

(4) [10 points] Let $W = \text{Span}\{v_1, \ldots, v_p\}$. Show that if $\mathbf{x}$ is orthogonal to v_j, for $1 \le j \le p$, then $\mathbf{x}$ is orthogonal to every vector in W.

Let $\vec{y}$ be an arbitrary vector in W. We will show $\vec{x} \cdot \vec{y} = 0$ and this will establish the result.

As $\vec{y}$ is in W, there exist scalars $c_1, \ldots, c_p$ so that

$\vec{y} = c_1 \vec{v}_1 + \ldots + c_p \vec{v}_p$.

We compute $\vec{x} \cdot \vec{y} = \vec{x} \cdot (c_1 \vec{v}_1 + \ldots + c_p \vec{v}_p)$ by a substitution

$$= \vec{x} \cdot c_1 \vec{v}_1 + \ldots + \vec{x} \cdot c_p \vec{v}_p$$

$$= c_1 (\vec{x} \cdot \vec{v}_1) + \ldots + c_p (\vec{x} \cdot \vec{v}_p)$$ by properties of dot product

$$= c_1 (0) + \ldots + c_p (0)$$ by the given, $\vec{x}$ is orthogonal to each $\vec{v}_j$.

$$= 0$$

Thus $\vec{x}$ is orthogonal to $\vec{y}$.

As $\vec{y}$ was arbitrary, $\vec{x}$ is orthogonal to each vector in W.

(Oops! Looks like I like this question given that I asked it before.)

(5) [10 points] Find an orthonormal basis for the column space of the given matrix. $B=\begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ Let $B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$.

It is easy to see that $\vec{b}_2 = 2\vec{b}_1$ and that $\vec{b}_3$ is not a scalar multiple of $\vec{b}_1$. So without doing any Gaussian Elimination, we see that $\{\vec{b}_1, \vec{b}_3\}$ is a basis for the column space of B.

Taking $\{\vec{b}_1, \vec{b}_3\}$ we construct an orthogonal basis via the Gram-Schmidt Process.

Thus $\vec{v}_1 = \vec{b}_1$.

$$\vec{v}_2 = \vec{b}_3 - \frac{\vec{b}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} - \frac{12}{12}\begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

So $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for Col B.

We normalize each vector to find an orthonormal basis.

$\|\vec{v}_1\| = \sqrt{1^2+3^2+1^2+1^2} = \sqrt{12}$, $\|\vec{v}_2\| = \sqrt{1^2+(-1)^2+1^2+1^2} = \sqrt{4} = 2$.

Thus $\left\{ \begin{bmatrix} 1/\sqrt{12} \\ 3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} \sqrt{12}/12 \\ 3\sqrt{12}/12 \\ \sqrt{12}/12 \\ \sqrt{12}/12 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \right\}$ is an orthonormal basis.

(6) [7 points] Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 1 & 6 \\ 1 & 8 \\ 1 & 2 \\ 2 & -8 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

To find the least-squares solution, we form the normal equations $A^TA\vec{x} = A^T\vec{b}$ and solve these.

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 6 & 8 & 2 & -8 \end{bmatrix}\begin{bmatrix} 1 & 6 \\ 1 & 8 \\ 1 & 2 \\ 2 & -8 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 168 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 6 & 8 & 2 & -8 \end{bmatrix}\begin{bmatrix} 7 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 44 \end{bmatrix}$$

So we consider $\left[\begin{array}{cc|c} 7 & 0 & 11 \\ 0 & 168 & 44 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 11/7 \\ 0 & 1 & 44/168 \end{array}\right]$.

Thus, $\hat{x} = \begin{bmatrix} 11/7 \\ 44/168 \end{bmatrix}$

Alternatively, if you observe that the columns of A are orthogonal, we may compute $\hat{b}$. $\hat{b} = \dfrac{\vec{b}\cdot\vec{a}_1}{\vec{a}_1\cdot\vec{a}_1}\vec{a}_1 + \dfrac{\vec{b}\cdot\vec{a}_2}{\vec{a}_2\cdot\vec{a}_2}\vec{a}_2 = \dfrac{11}{7}\vec{a}_1 + \dfrac{44}{168}\vec{a}_2$.

The weights in this linear combination of the a_i's give us $\hat{x}$, $\hat{x} = \begin{bmatrix} 11/7 \\ 44/168 \end{bmatrix}$.

(7) [3 points] Show how to find the least-squares error for the previous problem, but don't actually do so since the arithmetic is miserable to do.

To find the least squares error, we compute $\|\vec{b} - A\hat{x}\| =$

$$\left\| \begin{bmatrix} 7 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 6 \\ 1 & 8 \\ 1 & 2 \\ 2 & -8 \end{bmatrix}\begin{bmatrix} 11/7 \\ 44/168 \end{bmatrix} \right\|$$

(8) [10 points] Theorem 5 of Chapter 6 states the following: Let $\{\mathbf{u_1}, \ldots, \mathbf{u_p}\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u_1} + \cdots + c_p\mathbf{u_p}$$

are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u_j}}{\mathbf{u_j} \cdot \mathbf{u_j}}, \ (j = 1, \ldots, p).$$

The proof sketch is as follows: By the orthogonality of the set, we know $\mathbf{y} \cdot \mathbf{u_j} = c_1(\mathbf{u_j} \cdot \mathbf{u_j})$ for each j, where $1 \leq j \leq p$. We can now solve for c_j, *if* $(\mathbf{u_j} \cdot \mathbf{u_j})$ is not zero.

Question: Why can't this dot product equal zero?

As the set $\{\vec{u}_1, \ldots, \vec{u}_p\}$ is a basis, $\vec{u}_j$ is <u>not</u> the zero vector. As the only vector that when "dotted with itself" yields 0 is the zero vector, $\vec{u}_j \cdot \vec{y}$ cannot be zero.

To be clearer, <u>I</u> should have written "for a specific j".

Question: If the numerator for c_j in the expression given above does equal 0, then what can you say about $\vec{y}$ in relation to $\vec{u}_j$? And, what can you say about $\vec{y}$ in relation to the other vectors in the set?

I graded "generously".

One can say that $\vec{y}$ is orthogonal to $\vec{u}_j$.

Further, $\vec{y}$ is a linear combination of the other vectors in the set.

(Note that we're saying that $\vec{y}$ is orthogonal to <u>just</u> $\vec{u}_j$, we're <u>not</u> saying that $c_j = 0$ for <u>all</u> $1 \leq j \leq p$, just the one. If you read it as "all", then you should have concluded that $\vec{y} = \vec{0}$.)

As an example, let $W = \mathbb{R}^3$ and $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. Then if $\vec{y} \cdot \vec{u}_1 = 0$, so we're saying $j=1$, we would have $\vec{y}$ can be written as a linear combination of $\vec{e}_2$ and $\vec{e}_3$.

(9) [5 points] Given $\mathbf{u} \neq \mathbf{0}$ in $\mathbb{R}^n$, let $L = \text{Span}\{\mathbf{u}\}$. Show that the mapping $\mathbf{x} \mapsto \text{proj}_L \mathbf{x}$ is a linear transformation.

We show that the mapping preserves vector addition and scalar multiplication and so satisfies the definition of linear transformation.

Note that $\text{proj}_L \vec{x} = \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$.

Let $\vec{x}$ and $\vec{y}$ be any two vectors. Then

$$\text{proj}_L(\vec{x}+\vec{y}) = \frac{(\vec{x}+\vec{y}) \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{\vec{x} \cdot \vec{u} + \vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} + \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \text{proj}_L(\vec{x}) + \text{proj}_L(\vec{y}).$$

Let $\vec{x}$ be any vector and c any scalar. Then

$$\text{proj}_L(c\vec{x}) = \frac{(c\vec{x}) \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = c\frac{(\vec{x} \cdot \vec{u})}{\vec{u} \cdot \vec{u}} \vec{u} = c \, \text{proj}_L \vec{x}.$$

(10) [5 points] Suppose that a matrix A has two distinct eigenvalues, λ_1 and λ_2. Is it possible for a basis for the eigenspace corresponding to λ_1 to intersect in a non-trivial fashion with any basis for the eignespace corresponding to λ_2? Why?

No, it is not possible.

Suppose that $\vec{x}$ was in each basis. Then by the definition of eigenvector $A\vec{x} = \lambda_1 \vec{x}$ and $A\vec{x} = \lambda_2 \vec{x}$.

Thus, $\lambda_1 \vec{x} = \lambda_2 \vec{x}$ and so $\lambda_1 = \lambda_2$, a contradiction!

Thus the bases do not intersect.