LINEAR ALGEBRA
EXAM 3
SPRING 2013

Name:
Honor Code Statement:

Additional Honor Code Statement:

Signature:

Directions: Complete all problems. Justify all answers/solutions. Calculators, texts, and notes are not permitted. Please turn off all electronic devices. Additional blank white paper is available at the front of the room. Good luck!

(1) [5 points] Let T be a transformation in $\mathbb{R}^3$ that rotates points about some line through the origin. If A is the associated matrix, find an eigenvalue of A and describe the eigenspace that corresponds to this value.

(2) [5 points] Define **orthonormal set**.

Date: May 16, 2013.

(3) [20 points] Given the following matrix A, make the following computations.

$$A= \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

(a) Find the two distinct eigenvalues of A.

(b) Find a basis for each of the corresponding eigenvalues.

(c) Diagonalize A.

(d) Use this diagonallization to compute A^4.

(4) [10 points] Let $W = \text{Span}\{v_1, \ldots, v_p\}$. Show that if $\mathbf{x}$ is orthogonal to v_j, for $1 \leq j \leq p$, then $\mathbf{x}$ is orthogonal to every vector in W.

(5) [10 points] Find an orthonormal basis for the column space of the given matrix. $B=\begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}$

(6) [7 points] Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 1 & 6 \\ 1 & 8 \\ 1 & 2 \\ 2 & -8 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 7 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(7) [3 points] Show how to find the least-squares error for the previous problem, but don't actually do so since the arithmetic is miserable to do.

(8) [10 points] Theorem 5 of Chapter 6 states the following: Let $\{\mathbf{u_1}, \ldots, \mathbf{u_p}\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u_1} + \cdots + c_p\mathbf{u_p}$$

are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u_j}}{\mathbf{u_j} \cdot \mathbf{u_j}},\ (j = 1, \ldots, p).$$

The proof sketch is as follows: By the orthogonality of the set, we know $\mathbf{y} \cdot \mathbf{u_j} = c_1(\mathbf{u_j} \cdot \mathbf{u_j})$ for each j, where $1 \leq j \leq p$. We can now solve for c_j, *if* $(\mathbf{u_j} \cdot \mathbf{u_j})$ is not zero.

Question: Why can't this dot product equal zero?

Question: If the numerator for c_j in the expression given above does equal 0, then what can you say about y in relation to u_j? And, what can you say about y in relation to the other vectors in the set?

(9) [5 points] Given $\mathbf{u} \neq \mathbf{0}$ in $\mathbb{R}^n$, let $L = \text{Span}\{\mathbf{u}\}$. Show that the mapping $\mathbf{x} \mapsto \text{proj}_L \mathbf{x}$ is a linear transformation.

(10) [5 points] Suppose that a matrix A has two distinct eigenvalues, λ_1 and λ_2. Is it possible for a basis for the eigenspace corresponding to λ_1 to intersect in a non-trivial fashion with any basis for the eignespace corresponding to λ_2? Why?