LINEAR ALGEBRA
EXAM 3
FALL 2021

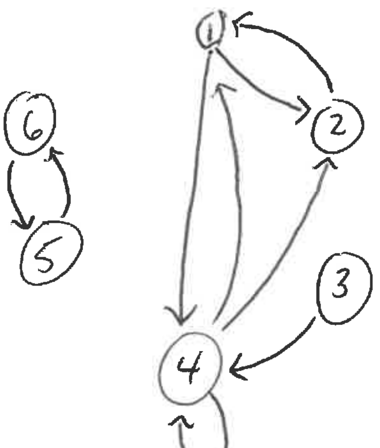
Name: Solution
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.
Signature: C.F. Gauss

70 points total
55 point avg.
15 point S.D.

Directions: Complete all problems. Justify all answers/solutions; answers without justifying calculations will not receive credit. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Good luck!

(1) [8 points] The following question is based upon the reading of *The 25 billion dollar eigenvector*. Consider the following link matrix, which is a column-stochastic matrix. Draw the corresponding web. Without doing any calculations, is there a page that will be ranked highest?

$$A=\begin{bmatrix} 0 & 1/2 & 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



Here's the web. Perhaps a little strange that page 4 voted for / linked to itself.

As the web is not connected the dimension of the eigenspace is greater than 1. Thus, there is not a unique ranking. So no page is ranked highest.

Date: December 18, 2021, 2–5pm.

(2) Let

$$A = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}.$$

(a) [5 points] Show that 3 is an eigenvalue of A by finding the characteristic equation of A and evaluating at 3.

The characteristic equation is given by $\det(A - \lambda I) = 0$.

So, $\det \begin{bmatrix} -2-\lambda & -4 & 2 \\ -2 & 1-\lambda & 2 \\ 4 & +2 & +5-\lambda \end{bmatrix} = 0$. If $\lambda = 3$, we get

$$\det \begin{bmatrix} -5 & -4 & 2 \\ -2 & -2 & 2 \\ 4 & +2 & 2 \end{bmatrix} = -5(-4-4) + 4(-4-8) + 2(-4+8)$$

$$= 40 - 48 + 8 = 0$$

Thus, $\lambda = 3$ is an eigenvalue.

(b) [5 points] Show that 3 is an eigenvalue of A by performing Gaussian elimination on the correct augmented matrix.

We're interested in $A\vec{x} = 3\vec{x}$ or non-trivial solutions to $[A - 3I \mid \vec{0}]$. So, perform G.E. on

$$\left[\begin{array}{ccc|c} -5 & -4 & 2 & 0 \\ -2 & -2 & 2 & 0 \\ +4 & 2 & 2 & 0 \end{array}\right] \underset{\substack{-\frac{1}{5}R1 \\ -\frac{1}{2}R2 \\ +\frac{1}{4}R3}}{\sim} \left[\begin{array}{ccc|c} 1 & 4/5 & -2/5 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1/2 & 1/2 & 0 \end{array}\right] \underset{\substack{-R1+R2 \\ -R1+R3}}{\sim} \left[\begin{array}{ccc|c} 1 & 4/5 & -2/5 & 0 \\ 0 & 1/5 & -3/5 & 0 \\ 0 & -3/10 & 9/10 & 0 \end{array}\right] \underset{+\frac{3}{2}R2+R3}{\sim} \left[\begin{array}{ccc|c} 1 & 4/5 & -2/5 & 0 \\ 0 & 1/5 & -3/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So we see that x_3 is free variable and the system is consistent. So, $\exists$ a non-trivial solution, i.e. $\lambda = 3$ is an eigenvalue.

(c) [3 points] Give a basis for the eigenspace associated with $\lambda = 3$.

Continuing w/ row reduction $(-4R2 + R1)$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1/5 & -3/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, $x_1 = -2x_3$

$\frac{1}{5}x_2 = +3/5\, x_3 \Rightarrow x_2 = 3x_3$

x_3 is free

So $\vec{x} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$ and this one vector serves as a basis.

(d) [2 points] State the dimension of the eigenspace associated with $\lambda = 3$.

Since there is one vector in the basis, the dimension is 1.

(e) [3 points] State how A *acts* on this eigenspace.

This eigenspace is a line through the origin in the direction of $\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$. ~~A~~ The matrix A acts on this line via dilation of a factor of 3.

(f) [4 points] The matrix A has two additional eigenvalues. These are -5 and 6. Is the matrix A diagonalizable? Why or why not?

Yes, the Diagonalization Theorem guarantees this, as does Theorems 6 and 7 of Chapter 5.

(g) [7 points] As -5 is an eigenvalue, there is an eigenvector associated to $\lambda = -5$. Call such a vector $\mathbf{v_1}$. Prove that $\mathbf{v_1}$ is NOT a scalar multiple of any vector found in the eigenspace associated with $\lambda = 3$.

Here is a nice proof that one student wrote:

"Suppose to the contrary that $\vec{v}_1$ is a scalar multiple of a vector found in the eigenspace associated with $\lambda = 3$.

Then we have $A\vec{v}_1 = 3\vec{v}_1$

Since $\vec{v}_1$ is an eigenvector associated to $\lambda = -5$, we have $A\vec{v}_1 = -5\vec{v}_1$.

So, by transitivity,

$$-5\vec{v}_1 = 3\vec{v}_1$$

Thus, $\vec{v}_1 = \vec{0}$. This is a contradiction since eigenvectors are non-zero vectors."

* Finding the basis for the eigenspace of $\lambda = -5$ via Gaussian Elimination was a computationally burdensome way to go though legitimate approach

(3) [4 points] Is there a vector in $\mathbb{R}^3$ with first two coordinates 3 and 4 with length less than 5? If yes, give such an example. If no, explain why not.

Let $\vec{x} = \begin{bmatrix} 3 \\ 4 \\ t \end{bmatrix} \in \mathbb{R}^3$.

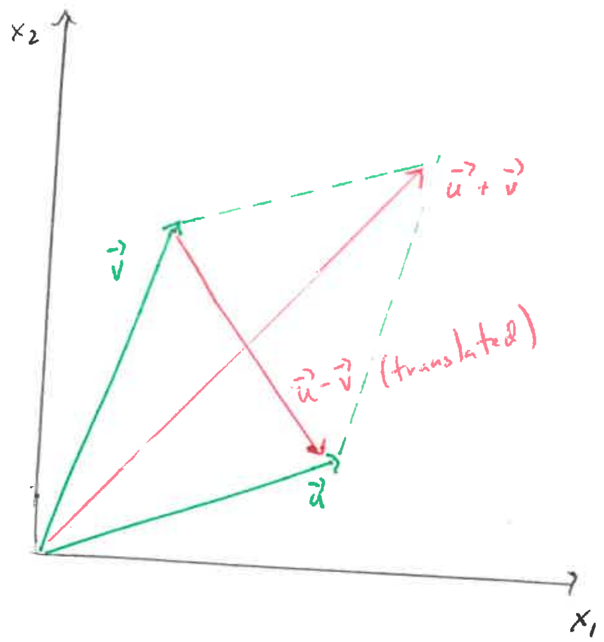
Then $\|\vec{x}\| = \sqrt{3^2 + 4^2 + t^2} = \sqrt{25 + t^2}$.

However, $\sqrt{25+t^2} \geq 5 \quad \forall\, t \in \mathbb{R}$.

So, there is no such vector.

(4) [6 points] Let **u** and **v** be two vectors in $\mathbb{R}^2$, each with positive coordinates. Draw a picture that illustrates vectors involved in the following identity. Then use this figure to explain what the identity is saying.

$$||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2||\mathbf{u}||^2 + 2||\mathbf{v}||^2$$



The identity says that the sum of the squared lengths of the diagonals of a parallelogram is equal to the sum of the squared lengths of its sides.

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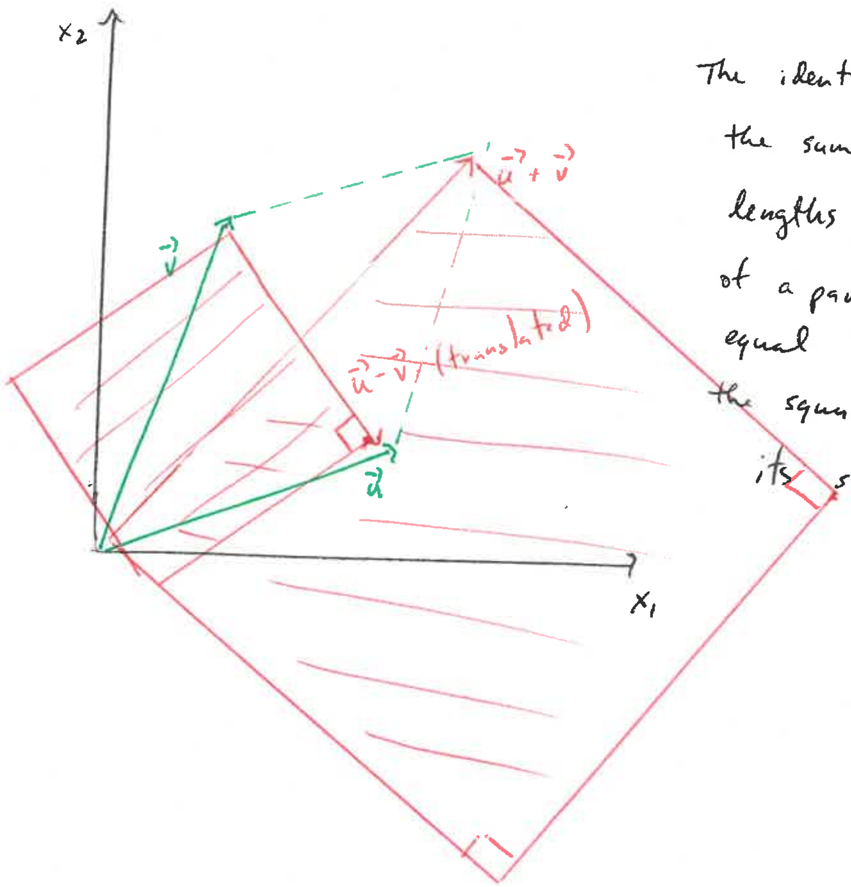

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left side is the area of the two red squares.

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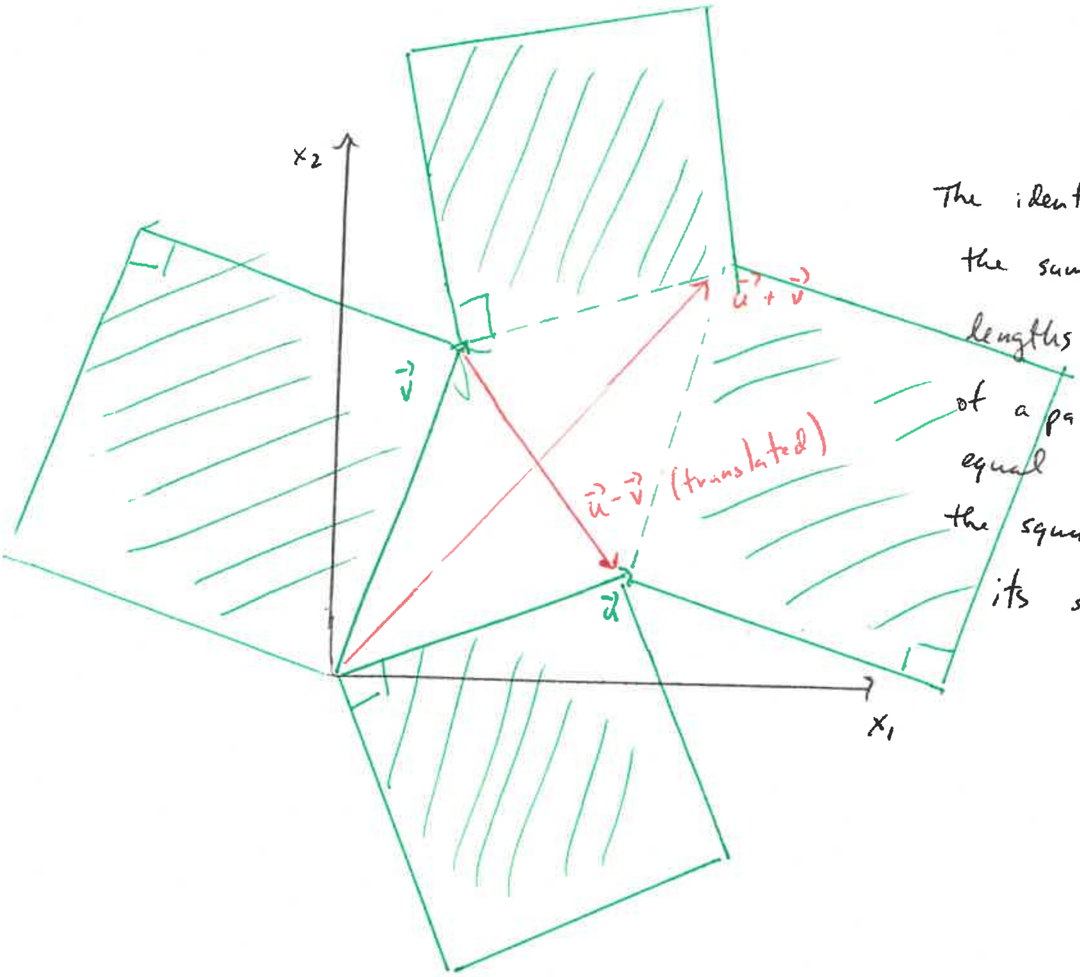

The identity says that the sum of the squared lengths of the diagonals of a parallelogram is equal to the sum of the squared lengths of its sides.

right side is the area of the four shaded green squares.

The identity says these areas are equal.

* Can one build a 3-D model of this?

(5) [5 points] Let $\mathbf{y}=\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ and $\mathbf{u}=\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

The Orthogonal Decomposition Theorem allows us to decompose $\vec{y}$ into $\hat{y}$ (its projection onto the line) and $\vec{z}$, a vector perpendicular to this line. The length of $\vec{z}$ is the distance we seek.

$$\hat{y} = \text{proj}_{\text{Span}\{\vec{u}\}} \vec{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{6}{10}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 9/5 \end{bmatrix}.$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} - \begin{bmatrix} -3/5 \\ 9/5 \end{bmatrix} = \begin{bmatrix} 18/5 \\ 6/5 \end{bmatrix}$$

$$\|\vec{z}\| = \sqrt{\left(\frac{18}{5}\right)^2 + \left(\frac{6}{5}\right)^2} = \sqrt{\frac{360}{25}} = \sqrt{\frac{36\cdot 10}{25}}$$

$$= \frac{6}{5}\sqrt{10}$$

(6) (a) [10 points] Apply the Gram-Schmidt Process to the following set of vectors in $\mathbb{R}^4$, $\mathbf{u_1} = (1,1,1,1), \mathbf{u_2} = (-1,4,4,-1), \mathbf{u_3} = (4,-2,2,0)$ which are a basis of some 3-dimensional subspace W. Call these new vectors $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$.

Applying the Gram-Schmidt Process:

we let $\vec{a}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Then $\vec{a}_2 = \vec{u}_2 - \operatorname{proj}_{\operatorname{Span}\{a_1\}} \vec{u}_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - \dfrac{\vec{u}_2 \cdot \vec{a}_1}{\vec{a}_1 \cdot \vec{a}_1} \vec{a}_1$

$$= \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - \frac{6}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix}$$

Then

$$\vec{a}_3 = \vec{u}_3 - \operatorname{proj}_{\operatorname{Span}\{\vec{a}_1, \vec{a}_2\}} \vec{u}_3$$

$$= \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{a}_1}{\vec{a}_1 \cdot \vec{a}_1} \vec{a}_1 - \frac{\vec{u}_3 \cdot \vec{a}_2}{\vec{a}_2 \cdot \vec{a}_2} \vec{a}_2$$

$$= \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-10}{25} \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}.$$

So, we get the orthogonal basis $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \right\}$

(b) [8 points] Let $A = [\mathbf{a_1}\ \mathbf{a_2}\ \mathbf{a_3}]$ and $\mathbf{b} = (0, 0, 0, 1)$. Find the least-squares solution of the equation $A\mathbf{x} = \mathbf{b}$.

Because the columns of A are orthogonal, the orthogonal projection of $\vec{b}$ onto Col A is given by

$$\hat{b} = \frac{\vec{b} \cdot \vec{a}_1}{\vec{a}_1 \cdot \vec{a}_1} \vec{a}_1 + \frac{\vec{b} \cdot \vec{a}_2}{\vec{a}_2 \cdot \vec{a}_2} \vec{a}_2 + \frac{\vec{b} \cdot \vec{a}_3}{\vec{a}_3 \cdot \vec{a}_3} \vec{a}_3$$

$$= \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{-5/2}{25} \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix} + \frac{-2}{16} \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

$$= \frac{1}{4} \vec{a}_1 - \frac{1}{10} \vec{a}_2 - \frac{1}{8} \vec{a}_3 .$$

Thus, $\hat{x} = \begin{bmatrix} 1/4 \\ -1/10 \\ -1/8 \end{bmatrix}$.