LINEAR ALGEBRA
EXAM 3
FALL 2021

Name:
Honor Code Statement:

Signature:

Directions: Complete all problems. Justify all answers/solutions; answers without justifying calculations will not receive credit. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Good luck!

(1) [8 points] The following question is based upon the reading of *The 25 billion dollar eigenvector*. Consider the following link matrix, which is a column-stochastic matrix. Draw the corresponding web. Without doing any calculations, is there a page that will be ranked highest?

$$A = \begin{bmatrix} 0 & 1/2 & 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Date: December 18, 2021, 2–5pm.

(2) Let

$$A = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}.$$

(a) [5 points] Show that 3 is an eigenvalue of A by finding the characteristic equation of A and evaluating at 3.

(b) [5 points] Show that 3 is an eigenvalue of A by performing Gaussian elimination on the correct augmented matrix.

(c) [3 points] Give a basis for the eigenspace associated with $\lambda = 3$.

(d) [2 points] State the dimension of the eigenspace associated with $\lambda = 3$.

(e) [3 points] State how A *acts* on this eigenspace.

(f) [4 points] The matrix A has two additional eigenvalues. These are -5 and 6. Is the matrix A diagonalizable? Why or why not?

(g) [7 points] As -5 is an eigenvalue, there is an eigenvector associated to $\lambda = -5$. Call such a vector $\mathbf{v_1}$. Prove that $\mathbf{v_1}$ is NOT a scalar multiple of any vector found in the eigenspace associated with $\lambda = 3$.

(3) [4 points] Is there a vector in $\mathbb{R}^3$ with first two coordinates 3 and 4 with length less than 5? If yes, give such an example. If no, explain why not.

(4) [6 points] Let **u** and **v** be two vectors in $\mathbb{R}^2$, each with positive coordinates. Draw a picture that illustrates vectors involved in the following identity. Then use this figure to explain what the identity is saying.

$$||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2||\mathbf{u}||^2 + 2||\mathbf{v}||^2$$

(5) [5 points] Let $\mathbf{y}=\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ and $\mathbf{u}=\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

(6) (a) [10 points] Apply the Gram-Schmidt Process to the following set of vectors in $\mathbb{R}^4$, $\mathbf{u_1} = (1, 1, 1, 1), \mathbf{u_2} = (-1, 4, 4, -1), \mathbf{u_3} = (4, -2, 2, 0)$ which are a basis of some 3-dimensional subspace W. Call these new vectors $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$.

(b) [8 points] Let $A = [\mathbf{a_1}\ \mathbf{a_2}\ \mathbf{a_3}]$ and $\mathbf{b} = (0, 0, 0, 1)$. Find the least-squares solution of the equation $A\mathbf{x} = \mathbf{b}$.