LINEAR ALGEBRA
EXAM 3
FALL 2020

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Signature: C.F. Gauss.

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Good luck!

75 points.

Average 64.5 pts.

standard deviation 12.25

Preparatory work: For this exam, you will use the following special vector - let's call it $\mathbf{s}$ - that is unique to you. The vector $\mathbf{s}$ is an element of $\mathbb{R}^3$, where the first entry is the number of letters in your first name, the second entry is the number of letters in your last name and the third entry is zero. (For example, John Schmitt would write the vector $\mathbf{s}=\begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix}$.) Write your $\mathbf{s}$ here.

It's often said that a pet (like a cat, dog or chicken) can make for a good *constant* companion. Write the name of your constant companion here and count the number of letters in that name. This number we will denote by c.

I have a cat named Peaches, so $c=7$ for me.

Date: December 11, 2020.

(1) [5 points] Give an example of a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that your special vector **s** is an eigenvector with eigenvalue c. Give another example of a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that your special vector **s** is **not** an eigenvector. Justify each.

An example of a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ so that $\vec{s}$ is an eigenvector with eigenvalue c is $T(\vec{x}) = c\vec{x}$

i.e. $\vec{x} \longmapsto A\vec{x}$ where $A = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$

as then $A\vec{s} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 4c \\ 7c \\ 0 \end{bmatrix} = c \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix} = c\vec{s}$.

(In fact, with this transformation every vector in $\mathbb{R}^3$ is an eigenvector w/ eigenvalue c.)

An example of a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ so that $\vec{s}$ is <u>not</u> an eigenvector is:

$\vec{x} \longmapsto A\vec{x}$ where $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Note that $A\vec{s} = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix}$ is <u>not</u> a scalar multiple of $\vec{s}$.

(2) [5 points] Let S be a $c \times c$ matrix with each entry of S equal to $\frac{1}{c}$. State what a steady-state vector for this matrix S would be and find one.

For me, $c = 7$ so

$$S = \underbrace{\begin{bmatrix} \frac{1}{7} & \cdots & 1/7 \\ \vdots & & \\ \frac{1}{7} & \cdots & 1/7 \end{bmatrix}}_{7 \times 7}$$

A steady state vector $\vec{x}$ would be an $\vec{x}$ s.t.

$S\vec{x} = \vec{x}$, i.e. an eigenvector of S w/ eigenvalue 1.

Let $\vec{x} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ then

$$S\vec{x} = 1 \begin{bmatrix} 1/7 \\ \vdots \\ 1/7 \end{bmatrix} + \cdots + 1 \begin{bmatrix} 1/7 \\ \vdots \\ 1/7 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

That is, the all-one vector is a steady state vector.

(3) [10 points] The following matrix A is **not** diagonalizable. Show that this is the case by first computing the eigenvalues of A and then by making some additional calculations.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

Since A is triangular, the eigenvalues of A are the entries on the main diagonal: 1, 2, 2.

For A to fail to be diagonalizable, the dimension of the eigenspace for $\lambda = 2$ must be 1 - this is according to Theorem 7 of Chapter 5. So, we show this via G.E. on $A\vec{x} = \lambda\vec{x}$; i.e. $(A - 2I)\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -3 & 5 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -3 & 5 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-1R1$, $R2 \leftrightarrow R3$ $3R1 + R2$, $-R1 + R3$

$\Rightarrow$ $x_1 = 0$

$x_2 = 0$

x_3 is free.

So $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix}$ where $s \in \mathbb{R}$.

Thus, the eigenspace has dimension 1.

Thus, A is <u>not</u> diagonalizable.

(4) [10 points] Construct an orthogonal basis for $\mathbb{R}^3$ containing your special vector $\mathbf{s}$. (Points are awarded for the simplicity of your solution.) Justify your solution.

My $\vec{s} = \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix}$.

The vector $\vec{v} = \begin{bmatrix} -7 \\ 4 \\ 0 \end{bmatrix}$ is orthogonal to $\vec{s}$ since $\vec{s} \cdot \vec{v} = -28 + 28 + 0 = 0$.

The vector $\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is orthogonal to both of these.

Recall that orthogonal sets are linearly independent, so $\{ \vec{s}, \vec{u}, \vec{v} \}$ is linearly independent.

Also, since this set contains 3 vectors and the dimension of $\mathbb{R}^3$ is three, the set is a basis, an orthogonal one!

where $S = \text{Span}\{\vec{s}, \vec{u}\}$.

(5) [5 points] The following two vectors are an orthogonal basis for a subspace S of $\mathbb{R}^3$, $\{\mathbf{s}, \mathbf{u}\}$ where $\mathbf{s}$ is your special vector and $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$.

Find the projection of $\mathbf{b}$ onto S, where $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, *and* the component of $\mathbf{b}$ orthogonal to S.

By the Orthogonal Decomposition Theorem,

$$\text{proj}_S \vec{b} = \frac{\vec{b} \cdot \vec{s}}{\vec{s} \cdot \vec{s}} \vec{s} + \frac{\vec{b} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$

$$= \frac{4}{16+49} \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix} + \frac{16}{64} \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$$

$$= \frac{4}{65} \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 16/65 \\ 28/65 \\ 2 \end{bmatrix} = \hat{b}$$

The orthogonal component is

$$\vec{b} - \text{proj}_S \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 16/65 \\ 28/65 \\ 2 \end{bmatrix} = \begin{bmatrix} 49/65 \\ -28/65 \\ 0 \end{bmatrix}$$

(6) [10 points] Use the normal equations to find the least squares solution of the following linear system $A\mathbf{x} = \mathbf{b}$ given by

$$2x_1 + 1x_2 = -5$$
$$-2x_1 = 8$$
$$2x_1 + 3x_2 = 1.$$

From the given, we have $A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$.

The normal equations are $A^T A \vec{x} = A^T \vec{b}$.

We form these and solve.

$$A^T A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$$

So, we perform G.E. on

$$\left[\begin{array}{cc|c} 12 & 8 & -24 \\ 8 & 10 & -2 \end{array}\right] \underset{\frac{1}{12}R1}{\sim} \left[\begin{array}{cc|c} 1 & 2/3 & -2 \\ 8 & 10 & -2 \end{array}\right] \underset{-8R1+R2}{\sim} \left[\begin{array}{cc|c} 1 & 2/3 & -2 \\ 0 & \frac{14}{3} & 14 \end{array}\right]$$

$$\underset{\frac{3}{14}R2}{\sim} \left[\begin{array}{cc|c} 1 & 2/3 & -2 \\ 0 & 1 & 3 \end{array}\right] \underset{-\frac{2}{3}R2+R1}{\sim} \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 3 \end{array}\right]$$

Thus, $\hat{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

(7) [5 points] (*Beware:* this is sort of a trick question.) Suppose that A is a 3×3 matrix the columns of which form a linearly independent set. If I were to ask for the least-squares error for the equation $A\mathbf{x} = \mathbf{s}$ (where $\mathbf{s}$ is your special vector), what is it? Explain in three sentences.

Since the three columns of A are a linearly independent set, they're also a basis. Thus, $\vec{s}$ is in the column space of A; i.e. $\exists$ a solution to $A\vec{x} = \vec{s}$. Thus, the least-squares error is zero.

(8) [10 points] Apply the Gram-Schmidt process to transform the following set of basis vectors into an orthogonal basis for the same subspace: $\mathbf{u_1} = (0, 2, 1, 0), \mathbf{u_2} = (1, -1, 0, 0), \mathbf{u_3} = (1, 0, 0, 1)$.

Let $\vec{v}_1 = \vec{u}_1 \, . = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

Then $\vec{v}_2 = \vec{u}_2 - \dfrac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} +1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \dfrac{(-2)}{5} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \frac{2}{5} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1/5 \\ 2/5 \\ 0 \end{bmatrix}$$

And, $\vec{v}_3 = \vec{u}_3 - \dfrac{\vec{u}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \dfrac{\vec{u}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - 0 \, \vec{v}_1 - \frac{1}{1 + \frac{1}{25} + \frac{4}{25}} \begin{bmatrix} 1 \\ -1/5 \\ 2/5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{25}{30} \begin{bmatrix} 1 \\ -1/5 \\ 2/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \\ -1/3 \\ 1 \end{bmatrix}$$

The orthogonal basis is $\{ \vec{v}_1 , \vec{v}_2 , \vec{v}_3 \}$.

(9) [5 points] **Google's PageRank Algorithm:** Google's PageRank algorithm seeks to rank webpages according to importance. Suppose that we form the $n \times n$ link matrix A for a web consisting of n webpages. This matrix is column-stochastic (i.e. each column sums to 1) and so there is a solution to $A\mathbf{x} = \mathbf{x}$. But suppose there are two solutions $\mathbf{x_1}$ and $\mathbf{x_2}$ which form a linearly independent set. What problem does this pose to ranking?

If there are two such solutions, then there would be two <u>distinct</u> rankings. Which ranking would we then use?

(10) [10 points] The Pythagorean Theorem states if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$. A proof of this theorem is given below.

PROOF:

$$\begin{aligned} ||\mathbf{u}+\mathbf{v}||^2 &= (\mathbf{u}+\mathbf{v})^T(\mathbf{u}+\mathbf{v}) && (1)\\ &= (\mathbf{u}^T+\mathbf{v}^T)(\mathbf{u}+\mathbf{v}) && (2)\\ &= \mathbf{u}^T\mathbf{u}+\mathbf{u}^T\mathbf{v}+\mathbf{v}^T\mathbf{u}+\mathbf{v}^T\mathbf{v} && (3)\\ &= ||\mathbf{u}||^2+||\mathbf{v}||^2. && (4)\end{aligned}$$

Of the four equality signs in the proof, circle the one which takes advantage of the assumption that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. In two or three sentences, use the definition of orthogonal to explain your choice.

Let $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$.

If $\vec{u}$ and $\vec{v}$ are orthogonal, then $\vec{u} \cdot \vec{v} = u_1v_1 + \ldots + u_nv_n = 0$.

Another way to say this is, $\vec{u}^T\vec{v} = u_1v_1 + \ldots + u_nv_n = 0$ and $\vec{v}^T\vec{u} = 0$.

Thus, these terms "drop out" and we are left with $\vec{u}^T\vec{u} = ||\vec{u}||^2 = (\vec{u} \cdot \vec{u})$ and $\vec{v}^T \cdot \vec{v} = \vec{v} \cdot \vec{v} = ||\vec{v}||^2$