LINEAR ALGEBRA
EXAM 3
FALL 2017

Name:
Honor Code Statement:

Signature:

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Please turn off all electronic devices – in fact, you shouldn't have any with you. Additional blank white paper is available at the front of the room – you are not permitted to use any other paper. Good luck!

85 points total.
avg. 78/85
wow!

(1) [5 points] Let $\mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathbf{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for $\mathbb{R}^2$. Find the change-of-coordinates matrix from $\mathbf{B}$ to $\mathbf{C}$.

$$\mathbf{b}_1 = \begin{bmatrix} 6 \\ -12 \end{bmatrix}$$
$$\mathbf{b}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$
$$\mathbf{c}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$
$$\mathbf{c}_2 = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

According to the discussion in the text following Theorem 15 of Section 4.7, to do this we perform Gaussian Elimination on

$$\left[\begin{array}{cc|cc} \vec{c}_1 & \vec{c}_2 & \vec{b}_1 & \vec{b}_2 \end{array}\right] = \left[\begin{array}{cc|cc} 4 & 3 & 6 & 4 \\ 2 & 9 & -12 & 2 \end{array}\right] \sim \underset{-\frac{1}{2}R1 + R2}{\left[\begin{array}{cc|cc} 4 & 3 & 6 & 4 \\ 0 & \frac{15}{2} & -15 & 0 \end{array}\right]}$$

$$\sim \underset{2/15\,R2}{\left[\begin{array}{cc|cc} 4 & 3 & 6 & 4 \\ 0 & 1 & -2 & 0 \end{array}\right]} \sim \underset{\frac{1}{4}R1}{\left[\begin{array}{cc|cc} 1 & 3/4 & 3/2 & 1 \\ 0 & 1 & -2 & 0 \end{array}\right]} \sim \underset{-3/4\,R2 + R1}{\left[\begin{array}{cc|cc} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & 0 \end{array}\right]}$$

Date: December 13, 2017.

Thus, $\underset{C \leftarrow B}{P} = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$.

(2) [20 points] Given the following matrix:

$$A = \begin{bmatrix} 10 & -8 \\ 2 & 2 \end{bmatrix}$$

(a) Determine the characteristic polynomial of A.

The characteristic polynomial is given by $\det(A - \lambda I)$.

So, $\det \begin{bmatrix} 10-\lambda & -8 \\ 2 & 2-\lambda \end{bmatrix} = (10-\lambda)(2-\lambda) + 16 = 20 - 12\lambda + \lambda^2 + 16$

$= 36 - 12\lambda + \lambda^2$

The characteristic polynomial is $p(\lambda) = \lambda^2 - 12\lambda + 36$.

(b) Determine the eigenvalues of A.

The eigenvalues ~~correspond~~ are equal to the roots of the characteristic polynomial. So we set $p(\lambda)$ equal to zero and solve for λ.

$$\lambda^2 - 12\lambda + 36 = 0$$

$$(\lambda - 6)(\lambda - 6) = 0$$

Thus, $\lambda = 6$ with multiplicity 2.

(c) For each of the eigenvalues determine a basis for the corresponding eigenspace.

To determine a basis for the eigenspace corresponding to $\lambda = 6$, we must solve $A\vec{x} = 6\vec{x}$, or $(A - 6I)\vec{x} = \vec{0}$. We use Gaussian Elimination on,

$$\left[\begin{array}{cc|c} 4 & -8 & 0 \\ 2 & -4 & 0 \end{array}\right] \sim \underset{\frac{1}{4}R1}{\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & -4 & 0 \end{array}\right]} \sim \underset{R2 - 2R1}{\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]}$$

Thus, $x_1 - 2x_2 = 0$ and x_2 is free. Thus, $\vec{x} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

and so $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace corresponding to $\lambda = 6$.

(d) Explain why, or why not, the matrix A is diagonalizable (you need not give the diagonalization if one exists).

Theorem 7 of Section 5.3 states that a square matrix is diagonalizable iff the dimension of each eigenspace equals the multiplicity of the corresponding eigenvalue. Here, the multiplicity of $\lambda = 6$ is 2, but the dimension of the eigenspace is 1. Thus, A is not diagonalizable.

(3) [10 points] A matrix A is diagonalizable, i.e. $A = PDP^{-1}$ where P and D are given below. Use this diagonalization to compute A^{10}. (You may leave the entries in the product in factored form.)

$P = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

As $A = PDP^{-1}$, $A^{10} = \underbrace{(PDP^{-1})(PDP^{-1})\ldots(PDP^{-1})}_{10 \text{ times}}$

$$A^{10} = PD^{10}P^{-1}$$

We first find P^{-1}:

$$P^{-1} = \frac{1}{3-4}\begin{bmatrix} 1 & -4 \\ -1 & 3 \end{bmatrix} = -1\begin{bmatrix} 1 & -4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$$

Thus,

$$A^{10} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2^{10} & 0 \\ 0 & 1^{10} \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot 2^{10} & 4 \\ 2^{10} & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -3\cdot 2^{10}+4 & 12\cdot 2^{10}-12 \\ -2^{10}+1 & 4\cdot 2^{10}-3 \end{bmatrix}$$

$$= \begin{bmatrix} -3068 & 12{,}276 \\ -1023 & 4093 \end{bmatrix}$$

(5) [10 points] With respect to the vectors of the previous problem, find the projection of $\mathbf{y}$ onto $W = Span\{\mathbf{u_1}, \mathbf{u_2}\}$ *and* the orthogonal component of $\mathbf{y}$.

We use the Orthogonal Decomposition Theorem, where $\vec{y} = \hat{y} + \vec{z}$ and $\hat{y} \in W$ and $\vec{z} \in W^{\perp}$

$$\hat{y} = \frac{\vec{y} \cdot \vec{u_1}}{\vec{u_1} \cdot \vec{u_1}} \vec{u_1} + \frac{\vec{y} \cdot \vec{u_2}}{\vec{u_2} \cdot \vec{u_2}} \vec{u_2}$$

$$= \frac{1}{3} \vec{u_1} + \frac{17}{9} \vec{u_2} = \frac{1}{3} \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix} + \frac{17}{9} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 43/9 \\ 25/9 \\ -17/9 \end{bmatrix}$$

and

$$\vec{z} = \vec{y} - \hat{y}$$

$$= \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 43/9 \\ 25/9 \\ -17/9 \end{bmatrix} = \begin{bmatrix} 2/9 \\ 2/9 \\ 8/9 \end{bmatrix}$$

(6) [10 points] Determine an orthogonal basis for the column space of B, where:

$$B=\begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 6 & 0 & 0 & 0 \end{bmatrix}$$

Recall that a basis for Col B is the set of pivot columns of B. To identify these, we use GE: $B \sim \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$. Thus the 1^{st}, 2^{nd} and 4^{th} columns are pivot columns and thus form a basis.

We apply the Gram-Schmidt process:

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 6 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 0 \\ 6 \end{bmatrix}}{\begin{bmatrix} 3 \\ 0 \\ 0 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 0 \\ 6 \end{bmatrix}} \begin{bmatrix} 3 \\ 0 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 0 \\ 6 \end{bmatrix}}{\begin{bmatrix} 3 \\ 0 \\ 0 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 0 \\ 6 \end{bmatrix}} \begin{bmatrix} 3 \\ 0 \\ 0 \\ 6 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{45} \begin{bmatrix} 3 \\ 0 \\ 0 \\ 6 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 0 \\ 1 \\ -2/5 \end{bmatrix}.$$

Now produce an orthonormal basis for $ColB$.

So, an orthogonal basis for Col B is $\left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4/5 \\ 0 \\ 1 \\ -2/5 \end{bmatrix} \right\}$

To produce an orthonormal basis, we normalize each vector to obtain

$$\left\{ \begin{bmatrix} 3/\sqrt{45} \\ 0 \\ 0 \\ 6/\sqrt{45} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4/\sqrt{45} \\ 0 \\ 5/\sqrt{45} \\ -2/\sqrt{45} \end{bmatrix} \right\}$$

(4) [10 points] *Without* doing Gaussian elimination, write y as a linear combination of the elements of the orthogonal basis S for $\mathbb{R}^3$, where $S = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$.

$$\mathbf{y} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$$

$$\mathbf{u_1} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$$

$$\mathbf{u_2} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$$\mathbf{u_3} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

We apply Theorem 5 of Chapter 6

$$\vec{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 + \frac{\vec{y} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} \vec{u}_3$$

$$\vec{y} = \frac{\begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}}{\begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}} \vec{u}_1 + \frac{\begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}} \vec{u}_2 + \frac{\begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}} \vec{u}_3$$

$$= \frac{6}{18} \vec{u}_1 + \frac{17}{9} \vec{u}_2 + \frac{4}{18} \vec{u}_3$$

$$\vec{y} = \frac{1}{3} \vec{u}_1 + \frac{17}{9} \vec{u}_2 + \frac{2}{9} \vec{u}_3 .$$

(7) [10 points] Find least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$ by constructing the normal equations for $\hat{\mathbf{x}}$ and solving for $\hat{\mathbf{x}}$.

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

The normal equations are

$$\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} \qquad A^T A \vec{x} = A^T \vec{b}$$

So, we compute $A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$

and $A^T \vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$

So we wish to solve $\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \hat{x} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$. We do so using G.E.

$$\left[\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 11 & 14 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 3 & 6 \\ 0 & 8 & 8 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

Thus, $\hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Compute the least-squares error associated with this solution.

Note $A\hat{x} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$.

The least-squares error is given by

$$\|\vec{b} - A\hat{x}\| = \left\| \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\| = \sqrt{1+1+4} = \sqrt{6}.$$

(8) [2 points per blank] **Theorem:** The eigenvalues of an upper triangular matrix are the entries on its main diagonal.

PROOF: Let A be an upper triangular matrix. As A is upper triangular, then $A - \lambda I$ has zeros below the main diagonal, has the same entries as A above the diagonal and the entries along the main diagonal are of the form $a_{ii} - \lambda$. The scalar λ is an eigenvalue of A if and only if the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a non-trivial solution, that is, if and only if the equation has a free variable. Because of the zero entries in $A - \lambda I$, it is easy to see that $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a free variable if and only if at least one of the entries on the diagonal of $A - \lambda I$ is zero. This happen if and only if λ equals one of the diagonal entries in A.

(9) [2 points per blank] Last week you were very kind when you offered to help find my lost eye glasses.