LINEAR ALGEBRA
EXAM 3
FALL 2017

Name:
Honor Code Statement:

Signature:

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Please turn off all electronic devices – in fact, you shouldn't have any with you. Additional blank white paper is available at the front of the room – you are not permitted to use any other paper. Good luck!

(1) [5 points] Let $\mathbf{B} = \{\mathbf{b_1}, \mathbf{b_2}\}$ and $\mathbf{C} = \{\mathbf{c_1}, \mathbf{c_2}\}$ be bases for $\mathbb{R}^2$. Find the change-of-coordinates matrix from $\mathbf{B}$ to $\mathbf{C}$.

$$\mathbf{b_1} = \begin{bmatrix} 6 \\ -12 \end{bmatrix}$$

$$\mathbf{b_2} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\mathbf{c_1} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\mathbf{c_2} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

Date: December 13, 2017.

(2) [20 points] Given the following matrix:

$$A= \begin{bmatrix} 10 & -8 \\ 2 & 2 \end{bmatrix}$$

(a) Determine the characteristic polynomial of A.

(b) Determine the eigenvalues of A.

(c) For each of the eigenvalues determine a basis for the corresponding eigenspace.

(d) Explain why, or why not, the matrix A is diagonalizable (you need not give the diagonalization if one exists).

(3) [10 points] A matrix A is diagonalizable, i.e. $A = PDP^{-1}$ where P and D are given below. Use this diagonalization to compute A^{10}. (You may leave the entries in the product in factored form.)

$P= \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$

$D= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(4) [10 points] *Without* doing Gaussian elimination, write y as a linear combination of the orthogonal basis S for $\mathbb{R}^3$, where $S = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$.

$$\mathbf{y} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$$

$$\mathbf{u_1} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$$

$$\mathbf{u_2} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$$\mathbf{u_3} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

(5) [10 points] With respect to the vectors of the previous problem, find the projection of $\mathbf{y}$ onto $W = Span\{\mathbf{u_1}, \mathbf{u_2}\}$ *and* the orthogonal component of $\mathbf{y}$.

(6) [10 points] Determine an orthogonal basis for the column space of B, where:

$$B = \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 6 & 0 & 0 & 0 \end{bmatrix}$$

Now produce an orthonormal basis for $ColB$.

(7) [10 points] Find least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$ by constructing the normal equations for $\hat{\mathbf{x}}$ and solving for $\hat{\mathbf{x}}$.

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

Compute the least-squares error associated with this solution.

(8) [2 points per blank] **Theorem:** The eigenvalues of an upper triangular matrix are the entries on its main diagonal.

PROOF: Let A be an upper triangular matrix. As A is upper triangular, then $A - \lambda I$ has zeros below the main diagonal, has the ______________ entries as A above the diagonal and the entries along the main diagonal are of the form ____ . The scalar λ is an eigenvalue of A if and only if the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a ____ , that is, if and only if the equation has a free variable. Because of the zero entries in $A - \lambda I$, it is easy to see that $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a free variable if and only if at least one of the entries on the diagonal of $A - \lambda I$ is ____________________________. This happen if and only if λ equals one of the diagonal entries in A.

(9) [2 points per blank] Last week you were very kind when you offered to help find my lost ____________________________.