LINEAR ALGEBRA
EXAM 3
FALL 2014

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam

Signature:

Total points: 95 points.

Avg: 75.9 points.

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Please turn off all electronic devices – in fact, you shouldn't have any with you. Additional blank white paper is available at the front of the room – you are not permitted to use any other paper. Good luck!

(1) [10 points] Suppose that a stranger approached you and gave you a set of five vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_5\}$, asking you to perform the Gram-Schmidt Process on the set. You do so, and in the third step of the process you compute $\mathbf{v}_3$ to be the zero vector. What would you then say to the stranger?

Recall that $\vec{v}_3$ is equal to $\vec{x}_3 - \text{proj}_{W_2} \vec{x}_3$, where $W_2 = \text{Span}\{\vec{v}_1, \vec{v}_2\} = \text{Span}\{\vec{x}_1, \vec{x}_2\}$. If $\vec{v}_3 = \vec{0}$, i.e. $\vec{0} = \vec{x}_3 - \text{proj}_{W_2} \vec{x}_3$, then $\vec{x}_3$ is in W_2. This means that $\vec{x}_3$ is a linear combination of $\vec{x}_1$ and $\vec{x}_2$. Thus the set given by the stranger is not linearly independent, and so not a basis. The Gram-Schmidt Process takes a basis for a non-zero subspace for W and outputs an orthogonal basis for W.

So, I'd politely tell the stranger to go away.

Date: December 9, 2014.

(2) [15 points] Find (1) the orthogonal projection of $\mathbf{b}$ onto Col A, (2) a least-squares solution of $A\mathbf{x} = \mathbf{b}$, and (3) the least-squares error.

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix}$$

and $\mathbf{b} = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(1) First notice that the columns of $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]$ form an orthogonal set, and so the set is a linearly independent set. This set of 3 vectors is a basis for Col A.

The Orthogonal Decomposition Theorem tells us that

$$\hat{b} = \frac{\vec{b}\cdot\vec{a}_1}{\vec{a}_1\cdot\vec{a}_1}\vec{a}_1 + \frac{\vec{b}\cdot\vec{a}_2}{\vec{a}_2\cdot\vec{a}_2}\vec{a}_2 + \frac{\vec{b}\cdot\vec{a}_3}{\vec{a}_3\cdot\vec{a}_3}\vec{a}_3$$

$$= \frac{36}{54}\vec{a}_1 + 0\,\vec{a}_2 + \frac{9}{27}\vec{a}_3 = \frac{2}{3}\vec{a}_1 + \frac{1}{3}\vec{a}_3 = \begin{bmatrix} 8/3 \\ 2/3 \\ 4 \\ 2/3 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ -5/3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \\ 4 \\ -1 \end{bmatrix}$$

(2) The least-squares solution of $A\vec{x} = \vec{b}$, is the vector $\hat{x}$ that solves $A\vec{x} = \hat{b}$. We've just done this! $\hat{x} = \begin{bmatrix} 2/3 \\ 0 \\ 1/3 \end{bmatrix}$.

(3) The least-squares error is $\|\vec{b} - A\hat{x}\| = \|\vec{b} - \hat{b}\| = \left\| \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 4 \\ -1 \end{bmatrix} \right\|$

$$= \left\| \begin{bmatrix} 6 \\ -1 \\ -4 \\ 1 \end{bmatrix} \right\|$$

$$= \sqrt{36 + 1 + 16 + 1}$$

$$= \sqrt{54} = 3\sqrt{6}$$

(3) [10 points] Let α and β be real numbers between 0 and 1. Let $P = \begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix}$. Find the steady-state vector of P, i.e. find a probability vector $\mathbf{q}$ satisfying $P\mathbf{q} = \mathbf{q}$.

Let us solve $P\vec{x} = \vec{x}$ for $\vec{x} \in \mathbb{R}^2$.

We obtain $P\vec{x} - I\vec{x} = \vec{0}$

$$(P - I)\vec{x} = \vec{0}$$

which as an augmented matrix can be written as

$$\left[\begin{array}{cc|c} -\alpha & \beta & 0 \\ \alpha & -\beta & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -\alpha & \beta & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} \alpha & -\beta & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So x_2 is a free variable and $\alpha x_1 = \beta x_2 \Rightarrow x_1 = \frac{\beta}{\alpha} x_2$

Thus, $\vec{x} = \begin{bmatrix} \beta/\alpha \, x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \beta/\alpha \\ 1 \end{bmatrix}$. So, $\begin{bmatrix} \beta/\alpha \\ 1 \end{bmatrix}$ solves $P\vec{x} = \vec{x}$

Recall that a probability vector must have its entries ~~po~~ nonnegative and summing to 1. Taking the entries in the found vector and summing them, we obtain $\frac{\beta}{\alpha} + 1 = \frac{\beta + \alpha}{\alpha}$, and we multiply the vector by the reciprocal of this to obtain

$$\frac{1}{\beta+\alpha/\alpha} \begin{bmatrix} \beta/\alpha \\ 1 \end{bmatrix} = \frac{\alpha}{\beta + \alpha} \begin{bmatrix} \beta/\alpha \\ 1 \end{bmatrix} = \begin{bmatrix} \beta/(\beta+\alpha) \\ \alpha/(\beta+\alpha) \end{bmatrix},$$

which is our steady-state vector.

(4) [10 points] Let us construct the companion matrix C_p of the following polynomial: $p = p(t) = -24 + 26t - 9t^2 + t^3 = (t-2)(t-3)(t-4)$.

$$C_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 24 & -26 & 9 \end{bmatrix}$$

Note that the entires of the last row of C_p correspond to coefficients of p. Find the characteristic polynomial of C_p. How does the characteristic polynomial of this matrix relate to the given polynomial and what does this imply about the eigenvalues of C_p?

The characteristic polynomial of C_p is given by

$$\det(C_p - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 24 & -26 & 9-\lambda \end{bmatrix}\right).$$

We'll do co-factor expansion down the 3rd column:

$$0 + (-1)^{2+3}(1)\begin{vmatrix} -\lambda & 1 \\ 24 & -26 \end{vmatrix} + (-1)^{3+3}(9-\lambda)\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix}$$

$$= -(26\lambda - 24) + (9-\lambda)\lambda^2$$

$$= -\lambda^3 + 9\lambda^2 - 26\lambda + 24.$$

Up to a change in the name of the variable, the two polynomials under consideration are opposites:

$$p(\lambda) = -\det(C_p - \lambda I).$$

Thus, the eigenvalues of C_p are the roots of p: 2, 3, and 4.

(Egads! Many of you thought that the roots of $-p(t)$ are different in sign to the roots of $p(t)$. They're not!)

(5) [10 points] The eigenvalues of the matrix that follows are 2 and 5. Use this to diagonalize the matrix.

$$D=\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

We find eigenvectors corresponding to each eigenvalue:

$\lambda = 2$: $D\vec{x} = 2\vec{x} \Rightarrow (D - 2I)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$ $x_1 = -x_2 - x_3$, x_2, x_3 free $\quad \vec{x} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix}$

$\vec{x} = x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$

Linearly independent eigenvectors corresponding to $\lambda = 2$ are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 5$: $(D - 5I)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & -3/2 & 3/2 & 0 \\ 0 & 3/2 & -3/2 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$ x_3 free, $x_1 = x_3$, $x_2 = x_3$

$\Rightarrow \vec{x} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Now We have a set of 3 linearly independent eigenvectors

to form $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $\tilde{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ Thus $D = P\tilde{D}P^{-1}$

One may find P^{-1} by performing G.E. on $[P \mid I]$. (see next sheet)

Here we find P^{-1}:

$[\,P \mid I\,] =$

$$\left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 2 & 1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -2/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & -1/3 & -1/3 & 2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 2/3 & -1/3 \\ 0 & 1 & 0 & -1/3 & -1/3 & 2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

$$\Rightarrow \quad P^{-1} = \begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Many students failed to read the directions here. First, you needn't tell me the statement is false (or true), since I told you it is ~~true~~ false. Second, you needed to provide a counter-example.

(6) [5 points each] Each of the following statements is false. Construct a counter-example for each to show that this is so. Justify your answer.

(a) The length of every vector is a positive number.

Vectors have non-negative length. The only non-negative number that is not positive is zero. So, what vector has length zero? The zero vector!

The zero vector is the only counterexample

(b) The sum of two eigenvectors of a matrix A is also an eigenvector of A.

Let us use the previous example: we choose one eigenvector from the eigenspace corresponding to $\lambda = 2$ and one from eigenspace corresponding to $\lambda = 5$ and sum them. $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

We show $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ is not an eigenvector of D:

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

. Note $\begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$ is not a scalar multiple of $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, thus $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ is not an eigenvector.

(c) Each eigenvalue of A is also an eigenvalue of A^2.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. As A is triangular, the eigenvalues of A are 1 and 2.

Now $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ is also triangular, with eigenvalues of 1 and 4.

(d) If a 4×4 matrix A has few than 4 distinct eigenvalues, then A is not diagonalizable.

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Let $A = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$. As A is triangular, its eigenvalues are the diagonal values, 8 in this case. So A has only 1 distinct eigenvalues, but is obviously diagonalizable: $A = I_4 A I_4$.

(e) If r is any scalar, then $||r\mathbf{v}|| = r||\mathbf{v}||$.

Let r be any negative scalar, say $r = -2$,
and let $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Then $r\vec{v} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$ and $||r\vec{v}|| = \sqrt{(-2)^2 + 0^2 + 0^2} = \sqrt{4} = 2$

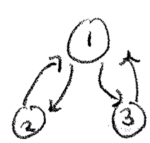
However, $r||\vec{v}|| = -2 \left|\left| \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right|\right| = -2\sqrt{1^2 + 0^2 + 0^2} = -2$.

(7) [10 points] Let S be an $n \times n$ matrix with each entry equal to $1/n$. Prove that if A is an $n \times n$ column-stochastic matrix and $0 \leq m \leq 1$, then $M = (1-m)A + mS$ is also a column-stochastic matrix.

As A is column stochastic, the entries in a given column sum to 1. Thus, the entries in a given column of $(1-m)A$ sum to $1-m$. Similarly, the entries in a given column of mS sum to m. Thus, the entries in a given column of M sum to $(1-m)+m = 1$. Thus, M is column stochastic.

(8) [5 points] Construct a 3-node web in which the PageRank algorithm explained in the article of Bryan and Leise would yield a unique ranking in which precisely two of the webpages have equal importance scores. Justify your response.

To have precisely two webpages with equal importance score, it would be beneficial to have "symmetry" in the network. To have a unique ranking (without having to resort to using matrix S), we ~~need~~ can make the network strongly connected, i.e. there is a directed path between any two nodes.

Here is such a network:  . The corresponding link matrix is

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}.$$

To see that all 3 nodes don't have the same importance score, we show that $\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ is not an eigenvector of A by direct computation:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/6 \\ 1/6 \end{bmatrix}.$$

Thus nodes 2 and 3 have equal importance score and the ranking is unique.