LINEAR ALGEBRA
EXAM 3
FALL 2014

Name:
Honor Code Statement:

Signature:

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Please turn off all electronic devices – in fact, you shouldn't have any with you. Additional blank white paper is available at the front of the room – you are not permitted to use any other paper. Good luck!

(1) [10 points] Suppose that a stranger approached you and gave you a set of five vectors $\{\mathbf{x_1}, \ldots, \mathbf{x_5}\}$, asking you to perform the Gram-Schmidt Process on the set. You do so, and in the third step of the process you compute $\mathbf{v_3}$ to be the zero vector. What would you then say to the stranger?

Date: December 9, 2014.

(2) [15 points] Find (1) the orthogonal projection of $\mathbf{b}$ onto Col A, (2) a least-squares solution of $A\mathbf{x} = \mathbf{b}$, and (3) the least-squares error.

$$A= \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix}$$

and $\mathbf{b}= \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(3) [10 points] Let α and β be real numbers between 0 and 1. Let $P = \begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix}$. Find the steady-state vector of P, i.e. find a probability vector $\mathbf{q}$ satisfying $P\mathbf{q} = \mathbf{q}$.

(4) [10 points] Let us construct the companion matrix C_p of the following polynomial: $p = p(t) = -24 + 26t - 9t^2 + t^3 = (t-2)(t-3)(t-4)$.

$$C_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 24 & -26 & 9 \end{bmatrix}$$

Note that the entires of the last row of C_p correspond to coefficients of p. Find the characteristic polynomial of C_p. How does the characteristic polynomial of this matrix relate to the given polynomial and what does this imply about the eigenvalues of C_p?

(5) [10 points] The eigenvalues of the matrix that follows are 2 and 5. Use this to diagonalize the matrix.

$$D = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

(6) [5 points each] Each of the following statements is false. Construct a counter-example for each to show that this is so. Justify your answer.

(a) The length of every vector is a positive number.

(b) The sum of two eigenvectors of a matrix A is also an eigenvector of A.

(c) Each eigenvalue of A is also an eigenvalue of A^2.

(d) If a 4×4 matrix A has few than 4 distinct eigenvalues, then A is not diagonalizable.

(e) If r is any scalar, then $||r\mathbf{v}|| = r||\mathbf{v}||$.

(7) [10 points] Let S be an $n \times n$ matrix with each entry equal to $1/n$. Prove that if A is an $n \times n$ column-stochastic matrix and $0 \leq m \leq 1$, then $M = (1 - m)A + mS$ is also a column-stochastic matrix.

(8) [5 points] Construct a 3-node web in which the PageRank algorithm explained in the article of Bryan and Leise would yield a unique ranking in which precisely two of the webpages have equal importance scores. Justify your response.