**LINEAR ALGEBRA**
**EXAM 3**
**FALL 2011**

**Name:**
**Honor Code Statement:**

**Signature:**

**Directions:** Complete all problems. Justify all answers/solutions. Each of the first two problems is worth 5 points. The third problem is worth 21 points (3 points for each part). Problem 8 is worth 15 points and the remaining problems are worth 10 each. Calculators, texts, and notes are not permitted. Please turn off all electronic devices. Additional blank white paper is available at the front of the room, in case I have not left you enough space to show your work. Good luck!

(1) State the Orthogonal Decomposition Theorem.

(2) Define **eigenvalue**.

*Date*: December 16, 2011.

(3) **True or False:** Mark each statement as True or False. If the statement is false, amend the statement in as few words as possible, and without simply negating the statement, so as to make a true statement or provide a counterexample.

(a) The sum of two eigenvectors of a matrix $A$ is also an eigenvector of $A$.

(b) Each eigenvalue of $A$ is also an eigenvalue of $A^2$.

(c) If $A$ contains a row of zeros, then 0 is an eigenvalue of $A$.

(d) The length of every vector is a positive number.

(e) If $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is an orthogonal set, then $\{7\mathbf{v_1}, 6\mathbf{v_2}, 5\mathbf{v_3}\}$ is an orthogonal set.

(f) If $\mathbf{x}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{x}$ must be orthogonal to $\mathbf{u} - \mathbf{v}$.

(g) A least squares solution of $A\mathbf{x} = \mathbf{b}$ is the vector $A\hat{\mathbf{x}}$ in Col $A$ closest to $\mathbf{b}$, so that $||\mathbf{b} - A\hat{\mathbf{x}}|| \leq ||\mathbf{b} - A\mathbf{x}||$ for all $\mathbf{x}$.

(4) Find the projection of $\mathbf{y}$ onto $W = Span\{\mathbf{u_1}, \mathbf{u_2}\}$ *and* the orthogonal component of $\mathbf{y}$.

$\mathbf{y} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

$\mathbf{u_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\mathbf{u_2} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

(5) With respect to the vectors in the previous problem, find the characteristic equation for the matrix $A = [\mathbf{u_1}\ \mathbf{u_2}]$. Then use this equation to find the eigenvalues of $A$.

(6) Find an orthonormal basis for the column space of the given matrix.

$$B = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$$

(7) Prove the following statement:

Let $\{\mathbf{u_1}, \ldots, \mathbf{u_p}\}$ be an orthogonal basis for a subspace $W$ of $\mathbb{R}^n$. For each $\mathbf{y}$ in $W$, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u_1} + \cdots + c_p\mathbf{u_p}$$

are given by $c_j = \frac{\mathbf{y}\cdot\mathbf{u_j}}{\mathbf{u_j}\cdot\mathbf{u_j}}$, $(j = 1, \ldots, p)$.

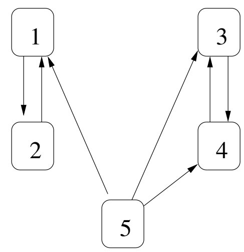


FIGURE 1. A web with five pages

(8) This problem references Figure 1

(a) Give the link matrix $A$, as defined in the article *The* \$*25,000,000,000 Eigenvector*.

(b) Determine the dimension of the eigenspace of $A$ corresponding to the eigenvalue of 1.

(c) The dimension of this eigenspace is either equal to one or it is not. What is the benefit with regard to PageRank if this eigenspace has dimension one?

(9) (a) Describe the steps taken to diagonalize a matrix.

(b) What is a computational benefit of having a diagonalization of a matrix $A$? (Recall that $A$ is diagonalizable if $A = PDP^{-1}$ where $D$ is a diagonal matrix.)