Linear Algebra
Exam 2 – Spring 2017

April 20, 2017

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators/texts/notes/cellphones are not permitted – the only permitted item is a pen or pencil.

1. [5 points] State the Spanning Set Theorem.

as given on page 210 of the text:

Let $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be a set in vector space V, and let $H = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$.

(1) If one of the vectors in S – say, $\vec{v}_k$ – is a linear combination of the others, then $S \setminus \{\vec{v}_k\}$ still spans H.

(2) If H is not the trivial subspace, then some subset of S is a basis for H.

2. [5 points] If the null space of an 8×7 matrix A is 5-dimensional, what is the dimension of the column space of A? what is the dimension of the row space of A? What theorem are you using to make these determinations?

By definition the dimension of the column space of A is the rank of A. By the Rank Theorem, $\text{Rank } A + \dim \text{Nul } A = n$, where n counts the columns of A. We are told that $n = 7$ and $\dim \text{Nul } A = 5$. [1] Thus, $\dim \text{Col } A = 2$. The Rank Theorem also tells us that ~~dim A~~ $\dim \text{Col } A = \dim \text{Row } A$. Thus, $\dim \text{Row } A = 2$.

3. [3 points each] Each of the following statements is false. Amend each statement in as few words or symbols as possible, without insertion or deletion of the word "not" (that is, without negating the conclusion), to make a true statement or explain why it's false, which can be done by pointing to a theorem or giving a counter-example.

(a) If A is a 3×3 matrix, then $det(5A) = 5detA$.

Change "5 det A" to "5^3 det A."

(b) $Nul\ A = \{\mathbf{0}\}$ if and only if the linear transformation is onto.

Change "onto" to "one-to-one".

(c) Let A be an $m \times n$ matrix. If $A\mathbf{x} = \mathbf{b}$ is consistent, then $Col\ A$ is $\mathbb{R}^n$.

Rewrite (two changes): If $A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$, then Col A is $\mathbb{R}^m$.

(d) The vector spaces $\mathbb{P}_3$ and $\mathbb{R}^3$ are isomorphic.

Change $\mathbb{P}_3$ to $\mathbb{P}_2$ or

change $\mathbb{R}^3$ to $\mathbb{R}^4$

(e) A plane in $\mathbb{R}^3$ is a two-dimensional subspace.

Insert, "a plane through the origin".

4. [5 points] Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for a vector space V, and suppose that $\mathbf{a}_1 = 2\mathbf{b}_1+4\mathbf{b}_2+8\mathbf{b}_3$, $\mathbf{a}_2 = 2\mathbf{b}_1+2\mathbf{b}_2+2\mathbf{b}_3$, and $\mathbf{a}_3 = \mathbf{b}_1+2\mathbf{b}_2+4\mathbf{b}_3$. If I asked you to find the change-of-coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$, I'd be asking for the impossible. Why?

One needs to notice that $\vec{a}_1 = 2\vec{a}_3$, and thus A is not a basis for V since the set is <u>not</u> linearly independent.

5. [5 points] The following is a subset of the vector space $\mathbb{R}^2$, but it is not a subspace. Why not? Justify any claim.

$$H = \{ \begin{bmatrix} 2 \\ 1+s \end{bmatrix} : s \in \mathbb{R}\}.$$

A subspace must contain the zero vector, but this set does not since the first coordinate of each vector in the set is 2.

<u>OR</u>. One may show that it is not closed under vector addition: let $\vec{u} = \begin{bmatrix} 2 \\ 1+0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1+1 \end{bmatrix}$

then $\vec{u}+\vec{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, which is not element of H.

<u>OR</u>. One may show that3 it is not closed under scalar multiplication. let $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in H$, and consider $4\vec{u} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \notin H$

Thus H fails to be closed under scalar multiplication.

6. [2 points per blank] **Fill in the blank**

Suppose that $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$, a subset of the vector space V, is linearly dependent. Then there exist scalars $c_1, \ldots, c_p$ not all zero with

$$c_1\mathbf{v_1} + \cdots + c_p\mathbf{v_p} = \mathbf{0}.$$

Since T is a linear transformation,

$$T(c_1\mathbf{v_1} + \cdots + c_p\mathbf{v_p}) = c_1T(\mathbf{v_1}) + \cdots + c_pT(\mathbf{v_p})$$

and $T(c_1\mathbf{v_1} + \cdots + c_p\mathbf{v_p}) = T(\mathbf{0}) = \mathbf{0}$.

Thus

$$c_1T(\mathbf{v_1}) + \cdots + c_pT(\mathbf{v_p}) = \mathbf{0}$$

and since not all of the c_i are zero, $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$ is also linearly dependent.

7. [5 points] The following two vectors do not form a basis for $\mathbb{R}^3$,

$\mathbf{b_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

and $\mathbf{b_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Determine a basis for $\mathbb{R}^3$ that contains these two vectors and justify that the set you give is indeed a basis.

We may find a vector not in the Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ and add it to this set. Claim $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in this span.

Proof:

$$\left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

As this system is inconsistent the claim is verified.

We now see that $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent, and by the IMT spans $\mathbb{R}^3$, and thus is a basis for $\mathbb{R}^3$, while containing the two original vectors.

8. Let $V = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$, where $a, b, c \in \mathbb{R}$.

[5 points] Use row operations to show that $det\ V$ equals $(b-a)(c-a)(c-b)$.

Adding a multiple of one row to another has no effect on the determinant of V. (Here V stands for Vandermonde matrix, woo-hoo!)

So,

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \sim \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix}$$

$-R1+R2$
$-R1+R3$

We may now do co-factor expansion down the first column.

$$\det V = 1 \cdot \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} = (b-a)(c^2-a^2) - (b^2-a^2)(c-a)$$

Recall that the difference of squares (c^2-a^2) factors as $(c-a)(c+a)$, and similarly for (b^2-a^2). Thus,

$$\det V = (b-a)(c-a)(c+a) - (b-a)(b+a)(c-a)$$
$$= (b-a)(c-a)\left[c+a-(b+a)\right] = (b-a)(c-a)(c-b).$$

[3 points]Under what conditions is V invertible?

By the IMT, V is invertible when $\det V \neq 0$.

Thus, so long as a, b and c are all distinct, V is invertible.

[6 points]

9. Let us consider the following three integers, $a = 2, b = 3$ and $c = 4$. Now, pick your favorite positive integer m. Let us try to determine if there exists a quadratic polynomial $f(x) = c_0 + c_1 x + c_2 x^2$ (i.e. a degree-two polynomial) such that $f(2) = 2^m, f(3) = 3^m$ and $f(4) = 4^m$. To do this, we want to see if the following system of equations has a solution.

$$c_0 + c_1 2 + c_2 2^2 = 2^m$$
$$c_0 + c_1 3 + c_2 3^2 = 3^m$$
$$c_0 + c_1 4 + c_2 4^2 = 4^m$$

Use the previous problem to verify that there is a solution. (Do not find the solution; we only care that it exists.)

The above system may be written as a matrix equation as follows.

$$\begin{bmatrix} 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ 1 & 4 & 4^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2^m \\ 3^m \\ 4^m \end{bmatrix}$$

As a, b, c are distinct, the above matrix is invertible and so the matrix equation (and thus the system) has a solution!