Linear Algebra
Exam 2 – Spring 2017

April 20, 2017

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators/texts/notes/cell-phones are not permitted – the only permitted item is a pen or pencil.

1. [5 points] State the Spanning Set Theorem.

2. [5 points] If the null space of an 8×7 matrix A is 5-dimensional, what is the dimension of the column space of A? what is the dimension of the row space of A? What theorem are you using to make these determinations?

3. [3 points each] Each of the following statements is false. Amend each statement in as few words or symbols as possible, without insertion or deletion of the word "not" (that is, without negating the conclusion), to make a true statement or explain why it's false, which can be done by pointing to a theorem or giving a counter-example.

   (a) If A is a 3×3 matrix, then $det(5A) = 5detA$.

   (b) $Nul\ A = \{\mathbf{0}\}$ if and only if the linear transformation is onto.

   (c) Let A be an $m \times n$ matrix. If $A\mathbf{x} = \mathbf{b}$ is consistent, then $Col\ A$ is $\mathbb{R}^n$.

   (d) The vector spaces $\mathbb{P}_3$ and $\mathbb{R}^3$ are isomorphic.

   (e) A plane in $\mathbb{R}^3$ is a two-dimensional subspace.

4. [5 points] Let $\mathcal{A} = \{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$ and $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$ be bases for a vector space V, and suppose that $\mathbf{a_1} = 2\mathbf{b_1}+4\mathbf{b_2}+8\mathbf{b_3}$, $\mathbf{a_2} = 2\mathbf{b_1}+2\mathbf{b_2}+2\mathbf{b_3}$, and $\mathbf{a_3} = \mathbf{b_1}+2\mathbf{b_2}+4\mathbf{b_3}$. If I asked you to find the change-of-coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$, I'd be asking for the impossible. Why?

5. [5 points] The following is a subset of the vector space $\mathbb{R}^2$, but it is not a subspace. Why not? Justify any claim.

$$H = \{ \begin{bmatrix} 2 \\ 1+s \end{bmatrix} : s \in \mathbb{R}\}.$$

6. [2 points per blank] **Fill in the blank**

   Suppose that $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$, a subset of the vector space V, is linearly dependent. Then there exist scalars $c_1, \ldots, c_p$ __________________________ with

   $$c_1\mathbf{v_1} + \cdots + c_p\mathbf{v_p} = \mathbf{0}.$$

   Since T is __________________________,

   $$T(c_1\mathbf{v_1} + \cdots + c_p\mathbf{v_p}) = c_1T(\mathbf{v_1}) + \cdots + c_pT(\mathbf{v_p})$$

   and $T(c_1\mathbf{v_1} + \cdots + c_p\mathbf{v_p}) = T(\mathbf{0}) = \mathbf{0}$.

   Thus

   $$c_1T(\mathbf{v_1}) + \cdots + c_pT(\mathbf{v_p}) = \mathbf{0}$$

   and since not all of the c_i are zero, __________________________ is also linearly dependent.

7. [5 points] The following two vectors do not form a basis for $\mathbb{R}^3$,

   $\mathbf{b_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

   and $\mathbf{b_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

   Determine a basis for $\mathbb{R}^3$ that contains these two vectors and justify that the set you give is indeed a basis.

8. Let $V = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$, where $a, b, c \in \mathbb{R}$.

[5 points] Use row operations to show that $det\ V$ equals $(b-a)(c-a)(c-b)$.

[3 points]Under what conditions is V invertible?

9. Let us consider the following three integers, $a = 2, b = 3$ and $c = 4$. Now, pick your favorite positive integer m. Let us try to determine if there exists a quadratic polynomial $f(x) = c_0 + c_1x + c_2x^2$ (i.e. a degree-two polynomial) such that $f(2) = 2^m, f(3) = 3^m$ and $f(4) = 4^m$. To do this, we want to see if the following system of equations has a solution.

$$
\begin{aligned}
c_0 + c_1 2 + c_2 2^2 &= 2^m \\
c_0 + c_1 3 + c_2 3^2 &= 3^m \\
c_0 + c_1 4 + c_2 4^2 &= 4^m
\end{aligned}
$$

Use the previous problem to verify that there is a solution. (Do not find the solution; we only care that it exists.)