Linear Algebra
Exam 2 - Spring 2014

April 24, 2014

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

60 points total

Average: 51.6

Standard Deviation: 6.2

Directions: Complete all problems. Justify all answers/solutions. Calculators/texts/notes are not permitted.

1. [5 points] State the Spanning Set Theorem.

As found on page 210 of the text:

Let $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ be a set in vectorspace V, and let $H = \text{Span}\{\vec{v}_1, \dots \vec{v}_p\}$

(i) If one of the vectors in S - say $\vec{v}_k$ - is a linear combination of the remaining vectors in S, then the set formed from S by removing $\vec{v}_k$ still spans H.

(ii) If $H \neq \{\vec{0}\}$, some subset of S is a basis for H.

2. [5 points] State **two** equivalent statements to the following: the $n \times n$ matrix A is invertible. Each statement should reference either the column space or null space of A.

Here are four possible statements:

(1) $\text{Col } A = \mathbb{R}^n$

(2) $\dim \text{Col } A = n = \text{rank } A$

(3) $\text{Nul } A = \{\vec{0}\}$

(4) $\dim \text{Nul } A = 0$.

3. Fill-in-the-blank - [2 points each]

(a) If A is invertible and has determinant 2, then the determinant of A^{-1} equals $1/2$.

This follows from Theorem 6 of Chapter 3: $\det(AB) = \det A \cdot \det B$, for A, B $n \times n$ matrices. As $AA^{-1} = I_n$ and $\det(I_n) = 1$, we have $\det A \cdot \det A^{-1} = 1 \Rightarrow \det A^{-1} = 1/2$, when $\det A = 2$.

(b) The column space of an $n \times n$ matrix A is all of $\mathbb{R}^n$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\vec{b}$ in $\mathbb{R}^n$.

(c) The pivot columns of a matrix A form a basis for Col A.

This is Theorem 6 of Section 4.2.

(d) Another way to say that the coordinate mapping is a one-to-one linear transformation from V onto $\mathbb{R}^n$ is to say it is an isomorphism.

See the definition of isomorphism on page 220.

(e) For an 8×9 matrix A, the dimension of the column space is 6. So, the dimension of the null space is 3.

This follows from the Rank Theorem, which states that dimension of column space plus dimension of the null space equals the number of columns of A.

4. Mark a true statement as true. Amend any false statement with as few changes as possible without simply inserting the word "not". [2 points each]

(a) A square matrix is invertible if and only if the determinant equals zero.

False, by the Invertible Matrix Theorem.

(b) In general, computing the determinant using row operations is faster than using co-factor expansion.

True. Co-factor Expansion on an $n \times n$ matrix takes $n!$ steps, as opposed to G.E., which takes order n^3 steps. Replace "faster" with "slower".

(c) For an $m \times n$ matrix A, the column space of A is a subspace of $\mathbb{R}^m$.

True. The column space is the set of all linear combinations of the columns of A, which have height m.

(d) Nul A $= \{ \mathbf{0} \}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has more than the trivial solution.

False. Replace "more than" by "only".

(e) If a vector space V has a basis with 8 vectors, then there may exist another basis with 7 vectors.

False. See Theorem 10 of Chapter 4. Replace 8 by 7, or 7 by 8.

5. [5 points] Find the determinant of the following matrix by first performing row replacements to create zeros in the first column. After this, do as you please.

$$A = \begin{bmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{bmatrix}$$

$$A \sim \underset{-2R1+R2}{\begin{bmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{bmatrix}} \sim \underset{3R1+R4}{\begin{bmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & -4 & 5 & 1 \\ 0 & 1 & -3 & 2 \end{bmatrix}} \sim \underset{-R2+R4}{\begin{bmatrix} 1 & -3 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & -4 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}} = B$$

As we have only done row-replacements to obtain B, $\det A = \det B$.

We can now compute the determinant of B using co-factor expansion across the 4^{th} row. So, we see $\det B = 0$. Thus, $\det A = 0$.

6. [5 points] Find an explicit description of Nul A by listing vectors that span the null space.

$$A = \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

To find such a description, we consider $A\vec{x} = \vec{0}$ and perform G.E. on the associated augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & -3 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -3 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

So x_2 and x_4 are free, $x_1 = 3x_2$ and $x_3 = 0$

$$\Rightarrow \vec{x} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, $\text{Nul}\, A = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

7. [5 points] Give an example of a 3×3 stochastic matrix.

A stochastic matrix is a square matrix whose column vectors are probability vectors, i.e. ~~vec~~ vectors w/ non-negative entries that sum to 1.

So, here's my example:

$$\begin{bmatrix} .2 & 1 & .5 \\ .4 & 0 & .5 \\ .4 & 0 & 0 \end{bmatrix}$$

8. [5 points] In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1-3t^2, 2+t-5t^2, 1+2t\}$ to the standard basis. Then write t^2 as a linear combination of the polynomials in $\mathcal{B}$.

We first map the vectors in $\mathcal{B}$ to vectors in $\mathbb{R}^3$. Doing so we obtain $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

Thus, the change of coordinates matrix is $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$.

To write t^2 as a linear combination of the polynomials in $\mathcal{B}$, we first express t^2 as a vector in $\mathbb{R}^3$: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Thus we wish to solve $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

We use Gaussian elimination - what else!

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -3 & -5 & 0 & 1 \end{array}\right] \sim \underset{3R1+R3}{\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 \end{array}\right]} \sim \underset{-R2+R3}{\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus, $t^2 = 3(1-3t^2) - 2(2+t-5t^2) + 1(1+2t)$.

9. [5 points] Show that the following set is not a subspace of $\mathbb{P}_2 : \{a + bt + 2t^2 | a, b \in \mathbb{R}\}$.

For the subset to be a subspace, it must contain the zero vector, be closed under vector addition and be closed under scaler multiplication.

We will show that it is ~~not~~ closed under vector addition, and thus show it is not a subspace.

$5 + 5t + 2t^2$ and $10 + 10t + 2t^2$ are both in the set.

Summing these, we obtain $15 + 15t + 4t^2$, which is not in the set since the coefficient of t^2 is not 2.

One can also show that it is not closed under scaler multiplication. It also doesn't include the zero vector.

10. **Fill-in-the-blank** [5 points] Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b_1} + \ldots + c_n\mathbf{b_n}.$$

PROOF: Since $\mathcal{B}$ spans V, there exist scalars such that $\mathbf{x} = c_1\mathbf{b_1} + \cdots + c_n\mathbf{b_n}$ holds. Suppose $\mathbf{x}$ also has the representation

$$\mathbf{x} = d_1\mathbf{b_1} + \cdots + d_n\mathbf{b_n}$$

for scalars $d_1, \ldots, d_n$. Then, subtracting, we have

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b_1} + \cdots + (c_n - d_n)\mathbf{b_n}.$$

Since $\mathcal{B}$ is linearly independent, the weights in this previous equation must all be zero. $\square$