Linear Algebra
Exam 2 - Spring 2014

April 24, 2014

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators/texts/notes are not permitted.

1. [5 points] State the Spanning Set Theorem.

2. [5 points] State **two** equivalent statements to the following: the $n \times n$ matrix A is invertible. Each statement should reference either the column space or null space of A.

3. Fill-in-the-blank - [2 points each]

(a) If A is invertible and has determinant 2, then the determinant of A^{-1} equals _____________ .

(b) The column space of an $n \times n$ matrix A is all of $\mathbb{R}^n$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has ____ in $\mathbb{R}^n$.

(c) The pivot columns of a matrix A form a basis for ______________________________________.

(d) Another way to say that the coordinate mapping is a one-to-one linear transformation from V onto $\mathbb{R}^n$ is to say it is ________________________________.

(e) For an 8×9 matrix A, the dimension of the column space is 6. So, the dimension of the null space is _____________.

4. Mark a true statement as true. Amend any false statement with as few changes as possible without simply inserting the word "not". [2 points each]

   (a) A square matrix is invertible if and only if the determinant equals zero.

   (b) In general, computing the determinant using row operations is faster than using co-factor expansion.

   (c) For an $m \times n$ matrix A, the column space of A is a subspace of $\mathbb{R}^m$.

   (d) Nul A = $\{ \mathbf{0} \}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has more than the trivial solution.

   (e) If a vector space V has a basis with 8 vectors, then there may exist another basis with 7 vectors.

5. [5 points] Find the determinant of the following matrix by first performing row replacements to create zeros in the first column. After this, do as you please.

$$A = \begin{bmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{bmatrix}$$

6. [5 points] Find an explicit description of Nul A by listing vectors that span the null space.
$A = \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

7. [5 points] Give an example of a 3×3 stochastic matrix.

8. [5 points] In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1-3t^2, 2+t-5t^2, 1+2t\}$ to the standard basis. Then write t^2 as a linear combination of the polynomials in $\mathcal{B}$.

9. [5 points] Show that the following set is not a subspace of $\mathbb{P}_2$: $\{a + bt + 2t^2 | a, b \in \mathbb{R}\}$.

10. **Fill-in-the-blank** [5 points] Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a ______________ set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b_1} + \ldots + c_n\mathbf{b_n}.$$

PROOF: Since $\mathcal{B}$ ______________ V, there exist scalars such that $\mathbf{x} = c_1\mathbf{b_1} + \cdots + c_n\mathbf{b_n}$ holds. Suppose $\mathbf{x}$ also has the representation

$$\mathbf{x} = d_1\mathbf{b_1} + \cdots + d_n\mathbf{b_n}$$

for scalars $d_1, \ldots, d_n$. Then, ______________ , we have

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b_1} + \cdots + (c_n - d_n)\mathbf{b_n}.$$

Since $\mathcal{B}$ is ______________ , the weights in this previous equation must all be ______________ . $\square$