Linear Algebra
Exam 2 - Spring 2013

April 18, 2013

Name:
Honor Code Statement:

Additional Honor Code Statement: I have not witnessed another violating the Honor Code.

SIGNATURE:

Directions: Complete all problems. Justify all answers/solutions. Calculators/texts/notes are not permitted, except as indicated in class on the first problem.

1. [10 points] Attach your solution to the end of the exam of the following problem. V and W are finite-dimensional vector spaces. T is a linear transformation from V to W. Let H be a nonzero subspace of V, and let $T(H)$ be the set of images of vectors in H. Then we know that $T(H)$ is a subspace of W by a previous exercise. Prove that $\dim T(H) \leq \dim(H)$.

2. [7 and 3 points] Compute the determinant of the following matrix. Justify your steps.

$$A = \begin{bmatrix} 4 & 1 & 1 & 2 \\ 6 & 1 & 3 & 2 \\ 0 & 1 & 5 & 0 \\ 0 & 4 & 1 & 0 \end{bmatrix}$$

Without doing Gaussian Elimination, state whether A has 4 pivot columns or not and justify your claim.

3. [5 points] Find a basis for the null space of the matrix A given in the previous problem.

4. [5 points] State the Basis Theorem.

5. [6 and 4 points] The set of vectors $\{\mathbf{b_1}, \mathbf{b_2}\}$, where $\mathbf{b_1}$ and $\mathbf{b_2}$ are given below, is not a basis for $\mathbb{R}^3$. Find a basis for $\mathbb{R}^3$ that contains this set. Justify your answer.

$$\mathbf{b_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{b_2} = \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}$$

Is it possible that another student in the class will have enlarged the set in a different manner than you did? Why? If the answer to the first question is yes, does your set share any commonalities other than $\mathbf{b_1}$ and $\mathbf{b_2}$ with another person's set ?

6. [5 points] Define **isomorphism**.

7. [5 points] Prove that the column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

8. Let V be a vector space. Let $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ be a basis of V and consider the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$. Show that the coordinate mapping is one-to-one.

TAKE-HOME Problem: This problem must be completed *on your own*. You may consult your text, but no other source. Please include the Honor Code statement and your signature upon turning this problem in. The problem is due on April 24 at the start of class.

9. [10 points] Consider the following set

$$V = \{(even, even), (even, odd), (odd, even), (odd, odd)\},$$

together with the rule that vector addition is done coordinate-wise, with $odd + odd = even, odd + even = even + odd = odd, even + even = even$, and scalar multiplication is also done coordinate-wise, with $even \cdot odd = even, even \cdot even = even, odd \cdot odd = odd, odd \cdot even = even$, and where scalars consist of *even* and *odd*. I claim that this set of four vectors, together with these rules, is a vector space.

(a) What is the zero vector (i.e. the additive identity) of this set?

(b) Give the additive inverse of each element in the set.

(c) Show (or argue) that the set is closed under vector addition.

(d) Which of the two scalars ‘plays the role’ of the scalar zero? Why?