Linear Algebra
Exam 2 - Fall 2023

November 9, 2023

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Avg: $\frac{43.4}{50}$

S.D. 8

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Please turn off all electronic devices – in fact, you shouldn't have any with you. Additional blank white paper is available at the front of the room – you are not permitted to use any other paper. Good luck!

1. [10 points] Compute the determinant of the following matrix first by co-factor expansion (across a row or down a column of your choosing), then second by using Gaussian Elimination. (Next page is blank to give space for your neatly presented computations.)

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

First we do co-factor expansion down column 4 due to the presence of many zeros.

$$\det A = 0 + 0 + 0 + (-1)^{4+4} \cdot 1 \cdot \det \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

For the 3×3 matrix, we do co-factor expansion down 3rd column

$$\det A = 1 \cdot 1 \cdot \left(0 + 0 + (-1)^{3+3} \cdot 2 \cdot \det \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \right)$$

$$= 2 \cdot \left(2 \cdot 4 - 1 \cdot 3 \right) = 2 \cdot 5 = 10$$

Now we compute the determinant using row operations, noting the effect of each row operation on the determinant.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -2 & -2 \end{bmatrix}$$

$R1 \leftrightarrow R4$

Interchange "negates"

$-3R1 + R2$

$-2R1 + R4$

Row replacements have no effect.

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix} = B$$

$R2 + R4$

Row replacements have <u>no</u> effect

$\frac{5}{2} R3 + R4$

Row replacement has no effect.

B is triangular. Its determinant is the product of the diagonal entries. Thus, $\det B = -10$. Thus, $\det A = -(-10) = 10$

2. [5 points] Based upon your answer to the previous question, is the set of column vectors contained in A linearly independent? Why or why not?

Since the determinant is non-zero, the Invertible Matrix Theorem tells us that the matrix is invertible and also its columns form a linearly independent set, an equivalent condition.

3. [5 points] Based upon your answer from that same problem about the determinant of A, is there some integer k such that the k−th power of A has determinant 0? That is, if we take higher and higher powers of A (like $A^2, A^3, A^4 \ldots$), is it possible that for some power k we have $det(A^k) = 0$?

By theorem 6 of Chapter 3, $\det(AA) = \det A \det A$

And so, $\det(A^k) = \underbrace{\det A \cdot \det A \ldots \det A}_{k \text{ times.}}$

Thus, $\det A^k = 10^k$. Thus, there is no power of k for which $\det A^k = 0$.

4. [10 points] Find a basis for the row space of A, the column space of A and the null space of A for the following matrix A.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Of course, Gaussian Elimination on A helps with this.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \cdots\cdots \text{ by calculations done on page 3 } \cdots \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix} = B$$

The rows containing pivots serve as a basis for Row A.

~~The~~ There are 4 pivot columns and so by ~~Ch. 4's~~ Theorem 6 of Chapter 4, the pivot columns of A (not B) form a basis of Col A.

As there are 4 pivots, further G.E. will yield the reduced echelon form to be $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Thus, the null space of A consists of just the $\vec{0}$ and the basis is empty.

5. [5 points] State the dimensions of each of the subspaces that you just found. Next, the mapping $\mathbf{x} \mapsto A\mathbf{x}$ has a domain and a co-domain. In which is the column space located? In which is the null space located?

dim Col $A = 4$

dim Row $A = 4$

dim Nul $A = 0$

Column space is in codomain.

Null space is in domain.

6. [5 points] The following set of vectors is not a basis for $\mathbb{R}^3$. Say why this is the case. Then find a basis for $\mathbb{R}^3$ that contains this set and say why the set you have constructed has the desired property.

$$\mathbf{b_1} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

By Theorem 10 of Chapter 4, any basis for $\mathbb{R}^3$ has 3 vectors. This set only has 2, thus cannot be a basis. So, we need a 3rd vector, say, $\vec{b}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

It must be that $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ forms a lin. ind. set and since it has 3 vectors, the Basis Theorem would imply it's a basis.

$$\begin{bmatrix} -1 & 1 & x \\ 2 & 0 & y \\ 0 & 2 & z \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & x \\ 0 & 2 & y+2x \\ 0 & 2 & z \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & x \\ 0 & 2 & y+2x \\ 0 & 0 & z-y-2x \end{bmatrix}$$

$2R1 + R2$ ~~$-2R$~~ $\qquad -R2 + R3$

So, if $z-y-2x \neq 0$, then there is a pivot in each column.

Let $x=1, y=1, z=1$, and this choice satisfies.

Thus, $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is a basis for $\mathbb{R}^3$.

7. [5 points] The following statements are false. Give a counter-example to each.

(a) If there exists a linearly dependent set $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ in V, then $dim V \leq p$.

A counter-example must consist of a vector space V and a linearly dependent set for which $\dim V > p$.

So, one possibility is:

$V = \mathbb{R}^3$, which has $\dim V = 3$

and the set
$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$ which is linearly dependent and has 2 vectors in it.

(b) A linearly independent set in a subspace H is a basis for H.

A counter-example must consist of a vector subspace H and a linearly independent set S which is <u>not</u> a basis for H.

Here is one possibility:

let $H = \text{Span}\{\vec{e}_1, \vec{e}_2\}$ for $\vec{e}_1, \vec{e}_2 \in \mathbb{R}^3$

and let $S = \{\vec{e}_1\}$.

8. [5 points] Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be bases for a vector space V, and suppose $\mathbf{f}_1 = 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3$, $\mathbf{f}_2 = 3\mathbf{d}_2 + \mathbf{d}_3$ and $\mathbf{f}_3 = -3\mathbf{d}_1 + 2\mathbf{d}_3$. Find the change-of-coordinates matrix from $\mathcal{F}$ to $\mathcal{D}$. Then find $[\mathbf{x}]_{\mathcal{D}}$ for $\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$.

From the given

$$[\vec{f}_1]_{\mathcal{D}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad [\vec{f}_2]_{\mathcal{D}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \quad [\vec{f}_3]_{\mathcal{D}} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

Thus,

$$\underset{\mathcal{D} \leftarrow \mathcal{F}}{P} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

So, if $[\vec{x}]_{\mathcal{F}} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$

Then

$$[\vec{x}]_{\mathcal{D}} = \underset{\mathcal{D} \leftarrow \mathcal{F}}{P} [\vec{x}]_{\mathcal{F}}$$

$$= \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}$$