Linear Algebra
Exam 2 - Fall 2021

November 11, 2021

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Please turn off all electronic devices – in fact, you shouldn't have any with you. Additional blank white paper is available at the front of the room – you are not permitted to use any other paper. Good luck!

1. [5 points] An **isomorphism** between two vector spaces V and W is a mapping that has three properties. List the three properties of this mapping.

2. [8 points] Use the fact that $det(AB) = det(A)det(B)$ to compute $det(C^5)$ where

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

3. [9 points] Suppose we are given the following equation involving block-partitioned matrices

$$\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix} \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Suppose that the partition of the matrices is conformable for block multiplication, that all sub-matrices are square and where I stands for the identity matrix.

Find formulas for the following matrices (in the suggested order) in terms of A, B and C:

(a) Find a formula for X.

(b) Find a formula for Z.

(c) FInd a formula for Y.

4. [5 points] Show that the following set is not a subspace of $\mathbb{R}^2$.

$W = \{ \begin{bmatrix} x \\ y \end{bmatrix} : xy = 0\}$

5. [5 points] Is the following vector **b** in the null space of C? Let

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

6. [2 points each] **Fill-in-the-blank**

(a) For an $m \times n$ matrix A, $Col A = \mathbb{R}^m$ if an only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution

____.

(b) The dimension of $Nul A$ is the

____ in the equation $A\mathbf{x} = \mathbf{0}$.

(c) In a p-dimensional vector space V, any linearly independent set of p vectors is automatically

____.

(d) For bases $\mathcal{B}$ and $\mathcal{C}$, the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ can be found by performing Gaussian elimination on the following augmented matrix

____.

7. [5 points] Suppose that H is a 1-dimensional subspace of the vector space $\mathbb{R}^3$. Consider a set of 3 vectors $\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$ that spans H. The Spanning Set Theorem states that if one of the vectors in S is a linear combination of the remaining vectors in S, then the set formed from S by removing it still spans H. Suppose that $\mathbf{b_2} = 2\mathbf{b_1}$ and that

$$\mathbf{b_1} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

If $Span\{\mathbf{b_3}\}$ is not H, what can you say about $\mathbf{b_3}$? Why does this not contradict the Spanning Set Theorem?

8. [8 points] Let $\mathbf{p_1} = 1 + t, \mathbf{p_2} = t + t^2$, and $\mathbf{p_3} = 1 + t + t^2$. Use coordinate vectors to show that these polynomials form a basis $\mathcal{B}$ for $\mathbb{P}_2$.

Find $\mathbf{q}$ in $\mathbb{P}_2$, given that $[\mathbf{q}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

9. [7 points] Suppose that a 6×8 matrix has four pivot columns. It is technically wrong to say that $Col A$ is *equal* to $\mathbf{R}^4$. What can you say? And, what vector space contains $Col A$?

How many vectors are in a basis for $Nul A$? Why?