Linear Algebra
Exam 2 - Fall 2020

November 12, 2020

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Please turn off all electronic devices – in fact, you shouldn't have any with you. Additional blank white paper is available at the front of the room – you are not permitted to use any other paper. Good luck!

1. [5 points] State the Rank Theorem.

2. [8 points] Compute the determinant of the following matrix. Justify your steps.

$$A = \begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix}$$

[2 points] Without making any further computation, do the columns of A form a linearly dependent set? Why or why not?

3. [10 points] Find an LU-decomposition of $A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$

4. [8 points] The following set of vectors is not a basis for $\mathbb{R}^3$. Find two distinct bases for $\mathbb{R}^3$ contained within this set. Justify your answer.

$$\{\mathbf{b_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{b_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b_3} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \mathbf{b_4} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$$

[2 points] How many distinct bases are contained within this set? Why?

5. [10 points] Consider the vector space of all polynomials in the variable t of degree at most 3; we denote this vector space as $\mathbb{P}_3$. Show that the following subset of $\mathbb{P}_3$ is a vector subspace of $\mathbb{P}_3$.

$$A = \{at^2 | a \in \mathbb{R}\}$$

6. [5 points each] Each of the following statements is false. Demonstrate that the statement is false by providing an example that meets the conditions of the hypothesis (the "if" part of the statement), but fails to meet the conclusion (the "then" part of the statement). That is, provide a counter-example to the statement.

   (a) If A is a 2×2 matrix and $A\mathbf{x} = \mathbf{b}$ is consistent, then $ColA$ is $\mathbb{R}^2$.

   (b) If we have a linearly independent set S in $\mathbb{R}^3$, then S is a basis for $\mathbb{R}^3$.

(c) If we take any plane P in $\mathbb{R}^3$, then P is a subspace of $\mathbb{R}^3$.

(d) If A is a 2×2 matrix and $\text{rank}(A) \geq 1$, then $Nul\ A = \{\mathbf{0}\}$.

7. [5 points] Describe how to test whether the following set of polynomials span $\mathbb{P}_2$. (You needn't make the determination.)

$$\{1 - 3t + 5t^2,\ -3 + 5t - 7t^2,\ -4 + 5t - 6t^2,\ 1 - t^2\}$$

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8. [5 points] Let $\mathcal{A} = \{\mathbf{a_1}, \mathbf{a_2}\}$ and $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}\}$ are bases for the vector space $\mathbb{R}^2$, where $\mathbf{a_1} = (1, 2), \mathbf{a_2} = (0, 1), \mathbf{b_1} = (1, 1), \mathbf{b_2} = (2, 1)$. Find the change of coordinates matrix from A to B. How do you know ahead of time that this matrix is invertible?