Linear Algebra
Exam 2 - Fall 2017

November 9, 2017

65 total points.

58 points 13 average.

Name: Solution Key

Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Directions: Complete all problems. Justify all answers/solutions. Calculators, cell-phones, texts, and notes are not permitted – the only permitted items to use are pens, pencils, rulers and erasers. Please turn off all electronic devices – in fact, you shouldn't have any with you. Additional blank white paper is available at the front of the room – you are not permitted to use any other paper. Good luck!

1. [10 points] The following is an LU-factorization of A. Use this factorization to solve the matrix equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is given below.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$$

To solve $A\vec{x} = \vec{b}$, we may instead solve $(LU)\vec{x} = \vec{b}$, or $L(U\vec{x}) = \vec{b}$. If we let $U\vec{x} = \vec{y}$, then we first solve $L\vec{y} = \vec{b}$ and then $U\vec{x} = \vec{y}$.

So we perform Gaussian Elimination on

$$\left[\, L \mid \vec{b}\, \right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & -5 \\ 3/2 & -5 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & -5 & 1 & 7 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -18 \end{array}\right] \quad \text{Thus, } \vec{y} = \begin{bmatrix} 0 \\ -5 \\ -18 \end{bmatrix}.$$

Now solve $U\vec{x} = \vec{y}$ using Gaussian Elimination: $\left[\begin{array}{ccc|c} 2 & -2 & 4 & 0 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & -6 & -18 \end{array}\right] \sim$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & -6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \text{Thus, } \vec{x} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$$

2. [8 points] Compute the determinant of the following matrix. Justify your steps.

$$G = \begin{bmatrix} 4 & 8 & 8 & 8 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 6 & 8 & 8 & 8 & 7 \\ 0 & 8 & 8 & 3 & 0 \\ 0 & 8 & 2 & 0 & 0 \end{bmatrix}$$

We first do co-factor expansion across the second row, so that

$$\det G = (-1)^{2+2} \cdot 1 \cdot \det \begin{bmatrix} 4 & 8 & 8 & 5 \\ 6 & 8 & 8 & 7 \\ 0 & 8 & 3 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

We next do ~~co-fal~~ co-factor expansion across the 4th row. to obtain,

$$\det G = +1 \cdot 1 \cdot (-1)^{4+2} \cdot 2 \cdot \det \begin{bmatrix} 4 & 8 & 5 \\ 6 & 8 & 7 \\ 0 & 3 & 0 \end{bmatrix}$$

Then we do co-factor expansion across the 3rd row to obtain,

$$\det G = 2 \cdot (-1)^{3+2} \cdot 3 \cdot \det \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$$

$$= -2 \cdot 3 \cdot (28 - 30) = 12$$

[2points] Is G invertible? Why?

According to the Invertible Matrix Theorem, as det G is not zero the matrix G is invertible.

3. [10 points] The following set of vectors is not a basis for $\mathbb{R}^3$. Find a basis for $\mathbb{R}^3$ that contains this set. Justify your answer.

$$\mathbf{b}_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

As $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a basis for $\mathbb{R}^3$, Theorem 10 of Chapter 4 states that every basis of this vector space must have 3 vectors. Also, the Basis Theorem implies that as $\dim \mathbb{R}^3 = 3$, any set of 3 linearly independent vectors will be a basis for $\mathbb{R}^3$. The given set of vectors is linearly independent ($\vec{b}_2$ is not a scalar multiple of $\vec{b}_1$). We can expand it by one vector to form a linearly independent set that is also spanning, hence a basis. So consider,

$$\left[\begin{array}{cc|c} \vec{b}_1 & \vec{b}_2 & \begin{matrix} x \\ y \\ z \end{matrix} \end{array}\right] = \left[\begin{array}{cc|c} 2 & 0 & x \\ 2 & 1 & y \\ 2 & 2 & z \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 0 & x \\ 0 & 1 & y-x \\ 0 & 2 & z-x \end{array}\right] \sim$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{x}{2} \\ 0 & 1 & y-x \\ 0 & 0 & z-x-2(y-x) \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & x/2 \\ 0 & 1 & y-x \\ 0 & 0 & x-2y+z \end{array}\right]$$

The system is consistent only when $x-2y+z=0$. Any choice of x, y, z s.t $x-2y+z \neq 0$ gives a vector not in Span $\{\vec{b}_1, \vec{b}_2\}$

Let $x=1$, $y=1$, $z=2$. Then $\vec{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is a basis for $\mathbb{R}^3$

4. [5 points] Let $H = \{\begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 = 0 \ or \ x^2 + y^2 = 1\}$. Is H a subspace of $\mathbb{R}^2$? Why or why not?

H is not a subspace as it is not closed under vector addition. We prove this. Consider $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which are elements of H as $1^2+0^2=1$ and $0^2+1^2=1$.

However $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which is not an element of H since $1^2+1^2 \neq 0$ and $1^2+1^2 \neq 1$.

5. [5 points] With regard to question 1, compute $\det A$ without computing A.

Theorem 6 of Chapter 3 tells us that

$\det A = \det L \det U$.

Also, we know that the determinant of a triangular matrix is the product of the diagonal entries.

Thus, $\det L = 1 \cdot 1 \cdot 1 = 1$ and $\det U = 2 \cdot (-2) \cdot (-6) = 24$

Thus $\det A = 1 \cdot 24 = 24$

6. [5 points] Suppose a 5×6 matrix A has four pivot columns.

(a) What is dim Nul A? Why?

As A has 6 columns 4 of which are pivot columns, there are 2 free variables to the system $A\vec{x} = \vec{0}$. The number of free variables corresponds to dim Nul A, thus dim Nul $A = 2$.

(b) Is Col $A = \mathbb{R}^4$? Why or why not?

Col A is not $\mathbb{R}^4$ as Col A is a subspace of $\mathbb{R}^5$. It follows from the Rank Theorem that dim Col $A = 4$ and so Col A is isomorphic to $\mathbb{R}^4$, but not equal to $\mathbb{R}^4$.

7. [5 points] State the Spanning Set Theorem.

See Theorem 5 of Chapter 4.

Let $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be a set of vectors in V and let $H = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$

(1) If one of the vectors - say, $\vec{v}_k$ - is a linear combination of the remaining vectors in S, then the set formed from S by removing $\vec{v}_k$ still spans H.

(2) If $H \neq \{\vec{0}\}$, some subset of S is a basis for H.

8. [5 points] The following matrix is a block partitioned matrix. Show/prove that the matrix has an inverse AND determine this inverse based upon the block entries.

$$\begin{bmatrix} I_p & 0 \\ A & I_q \end{bmatrix}$$

This square matrix is invertible since it has a pivot in every row/column.

The inverse of the matrix has the form $\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$

and satisfies $\begin{bmatrix} I_p & 0 \\ A & I_q \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}$

Computing the product and equating entries, we obtain:

$$\left.\begin{aligned} I_p B_{11} + 0 B_{21} &= I_p \\ I_p B_{12} + 0 B_{22} &= 0 \\ A B_{11} + I_q B_{21} &= 0 \\ A B_{12} + I_q B_{22} &= I_q \end{aligned}\right\} \Rightarrow \begin{aligned} B_{11} &= I_p \\ B_{12} &= 0 \\ A B_{11} + B_{21} &= 0 \\ A B_{12} + B_{22} &= I_q \end{aligned}$$

Knowing $B_{11} = I_p$ and $B_{12} = 0$, we may substitute into the latter equations $A + B_{21} = 0$ and $A \cdot 0 + B_{22} = I_q$, that is, $B_{21} = -A$ and $B_{22} = I_q$.

Thus, the inverse is $\begin{bmatrix} I_p & 0 \\ -A & I_q \end{bmatrix}$

9. **Fill-in-the-blank** [2 points for each blank] Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a one-to-one linear transformation. Let us determine the dimension of the range of T.

Let A be the $m \times n$ standard matrix of T. As T is one-to-one, the columns of A are linearly independent. So, we can conclude that the dimension of the null space of A equals 0. By the Rank Theorem, $\dim \mathrm{Col} A = n - 0 = n$, which is the number of columns of matrix A. As the range of T is Col A, the dimension of the range of T is n.