Linear Algebra
Exam 2 – Fall 2014

November 6, 2014

63 points total.

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this exam.

Directions: Complete all problems. Justify all answers/solutions. Calculators/texts/notes/cell-phones are not permitted – the only permitted item is a pen or pencil. Each problem is worth 5 points unless otherwise indicated.

1. State the Basis Theorem.

Let V be a p-dimensional vector space, $p \geq 1$.

- Any linearly independent set of exactly p elements in V is automatically a basis for V.
- Any set of exactly p elements that spans V is automatically a basis for V.

(see page 227 of text).

2. Let A be a 5×4 matrix. Suppose that the associated linear system $A\mathbf{x} = \mathbf{b}$ has one free variable. Does A have full rank?

No, A does not have full rank. Full rank for a matrix with four columns would be rank 4. However, rank is determined by the 1 number of pivot columns, which in this case is 3.

3. Find the coordinate vector of $\mathbf{x} = \begin{pmatrix} -1 \\ -6 \end{pmatrix}$ relative to the basis $\mathbf{b_1} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, $\mathbf{b_2} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

We seek the real #s c_1, c_2 such that

$$c_1 \vec{b}_1 + c_2 \vec{b}_2 = \vec{x}.$$

We ~~do~~ find these via Gaussian Elimination.

$$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ -4 & -3 & -6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 5 & -10 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

Thus, $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is the coordinate vector of $\vec{x}$ relative to the basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$.

4. Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for a vector space V, and suppose that $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2, \mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, and $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$. Find the change-of-coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$.

Theorem 15 of Chapter 4 tells us that $\underset{\mathcal{B} \leftarrow \mathcal{A}}{P}$ has columns that are the $\mathcal{B}$-coordinate vectors of the vectors in the basis $\mathcal{A}$. That is, $\underset{\mathcal{B} \leftarrow \mathcal{A}}{P} = \left[[\vec{a}_1]_{\mathcal{B}} \cdots [\vec{a}_n]_{\mathcal{B}} \right]$.

Thus

$$\underset{\mathcal{B} \leftarrow \mathcal{A}}{P} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

5. Compare the following two numbers:

   - the number of multiplications needed in a calculation of the determinant of a 6×6 matrix via co-factor expansion, and
   - the number of different ways for six people to finish a race (where ties are not possible).

To compute such a determinant in this manner, we must compute the determinant of six 5×5 matrices. Each of the 5×5 matrices requires one to compute 5 4×4 determinants. And, so on. ~~To compute~~ The number of 2×2 determinants we have to compute at the end of this is : $6\cdot5\cdot4\cdot3$ and each such 2×2 determinant requires two multiplications. Thus, we have $6\cdot5\cdot4\cdot3\cdot2 = 6!$ multiplications coming from these 2×2 determinants - and this is before we multiply by the corresponding weights. (See numerical note on p.167)

The number of ways for 6 people to finish a race is $6!$ via multiplication of choices. Thus, the # of multiplications are roughly equal

6. Give two 2×2 matrices A and B such that $\det(A + B) \neq \det(A) + \det(B)$. Justify your answer.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, with $\det A = 1$

Let $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, with $\det B = 0$

Thus $A + B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ with $\det(A+B) = 3$, but $\det A + \det B = 1$.

7. Compute the determinant of the following matrix.

$$A = \begin{bmatrix} 2 & -4 & 10 & 0 \\ 4 & 0 & 8 & -2 \\ 6 & 2 & 0 & 14 \\ 0 & 8 & -4 & 0 \end{bmatrix}$$

who chose this problem?!

Option A We'll do co-factor expansion down column 4.

$$\det A = (-1)^{2+4}(-2)\begin{vmatrix} 2 & -4 & 10 \\ 6 & 2 & 0 \\ 0 & 8 & -4 \end{vmatrix} + (-1)^{3+4}(14)\begin{vmatrix} 2 & -4 & 10 \\ 4 & 0 & 8 \\ 0 & 8 & -4 \end{vmatrix}$$

$$= -2\left((-1)^{3+2}\cdot 8 \cdot \begin{vmatrix} 2 & 10 \\ 6 & 0 \end{vmatrix} + (-1)^{3+3}(-4)\begin{vmatrix} 2 & -4 \\ 6 & 2 \end{vmatrix} \right)$$

$$- 14\left((-1)^{3+2}\cdot 8 \cdot \begin{vmatrix} 2 & 10 \\ 4 & 8 \end{vmatrix} + (-1)^{3+3}(-4)\begin{vmatrix} 2 & -4 \\ 4 & 0 \end{vmatrix} \right)$$

$$= -2\left(-8\cdot(-60) - 4(28)\right) - 14\left(-8(-24) - 4(16)\right)$$

$$= -2(480 - 112) - 14(192 - 64)$$

$$= -2(368) - 14(128)$$

$$= -736 - 1792$$

$$= -(736 + 1792)$$

$$= -2528$$

$$\begin{array}{r} 128 \\ \times\ 14 \\ \hline 512 \\ 128 \\ \hline 1792 \end{array}$$

Option B First we perform Gaussian Elimination, using only row replacements which do not alter the determinant.

$$A \sim \begin{bmatrix} 2 & -4 & 10 & 0 \\ 0 & 8 & -12 & -2 \\ 0 & 14 & -30 & 14 \\ 0 & 8 & -4 & 0 \end{bmatrix} = B$$

So $\det A = \det B = (-1)^{1+1}\cdot 2 \cdot \begin{vmatrix} 8 & -12 & -2 \\ 14 & -30 & 14 \\ 8 & -4 & 0 \end{vmatrix}$

$$= 2\left(8\begin{vmatrix} -12 & -2 \\ -30 & 14 \end{vmatrix} + 4\begin{vmatrix} 8 & -2 \\ 14 & 14 \end{vmatrix} \right)$$

$$= 2\left(8(-168 - 60) + 4(112 + 28)\right)$$

$$= 2\left(8(-228) + 560\right) = 2(-1824 + 560) = 2(-1264) = -2528$$

8. Each of the following statements ~~are~~ is false. Give a counterexample for each in order to demonstrate that the statement is indeed false. Justify each response.

(a) If the determinant of the 2×2 matrix A is zero, then either the rows are the same, the columns are the same or at least one row or column is all zeros.

Let A have columns that form a linearly dependent set, then by the IMT det A will equal zero. We can find a linearly dependent set w/o the vectors being identical, containing the zero vector, and managing to avoid the row condition. Here's an option: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, then $\det A = 4 - 4 = 0$.

(b) A subset H of the vector space $\mathbb{R}^3$ is a sub-space of $\mathbb{R}^3$ if the zero vector is in H.

Let $H = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ be a sub<u>set</u>. H is not closed under vector addition. For example, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$, but $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \notin H$. Thus, H is a sub<u>set</u> containing the zero vector but it is not a subspace.

(c) If $H = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} \subseteq \mathbb{R}^4$, then $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for H.

We need not have $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ forming a linearly independent set.

Thus, $\vec{b}_1 = \vec{b}_2 = \vec{b}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ spans H, but it is not a basis for H.

9. Show that if A is invertible that $\det(A^{-1}) = \frac{1}{\det(A)}$.
Hint: for $n \times n$ matrices A and B, we have $\det(AB) = \det(A)\det(B)$.

As A is invertible, $\det A \neq 0$ by the IMT.

Recall that $AA^{-1} = I_n$, and $\det(I_n) = 1$.

Thus, using the hint, we have $\det(AA^{-1}) = \det(A)\det(A^{-1}) = 1$.

Thus, $\det A^{-1} = \frac{1}{\det A}$, and this is defined since $\det A \neq 0$.

$\boxtimes$

10. Theorem 10 of Chapter 4 states: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Let $\mathcal{B}_1$ be a basis of n vectors and $\mathcal{B}_2$ any other basis of V.

When we proved this theorem, there were some questions about why the following statement was true: Since $\mathcal{B}_2$ is a basis and $\mathcal{B}_1$ is linearly independent, $\mathcal{B}_2$ has at least n vectors.

Let us expound on this statement. **Fill in the blanks – 2 points each.**

We need to show that $\mathcal{B}_2$ has at least n vectors. We have that $\mathcal{B}_1 = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ is a linearly independent set with n vectors and that $\mathcal{B}_2 = \{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is a spanning set with, say, p vectors. As $\mathcal{B}_1$ is a linearly independent set, the equation

$$x_1\mathbf{b_1} + \cdots + x_n\mathbf{b_n} = \mathbf{0}$$

has only the trivial solution. As $\mathcal{B}_2$ is a spanning set, each $\mathbf{b_i}$ is a linear combination of the $\mathbf{v_j}$'s. Thus, we can write the above equation in terms of the $\mathbf{v_j}$'s:

$$x_1(c_{1,1}\mathbf{v_1} + \cdots + c_{1,p}\mathbf{v_p}) + \cdots + x_n(c_{n,1}\mathbf{v_1} + \cdots + c_{n,p}\mathbf{v_p}) = \mathbf{0}.$$

The left side of this equation is a linear combination of the vectors in $\mathcal{B}_2$, while the right side is $\mathbf{0}$. As $\mathcal{B}_2$ is a linearly independent set, the only solution to this vector equation is the trivial solution. Note however that this equation gives rise to a linear system:

$$c_{1,1}x_1 + \cdots + c_{n,1}x_n = 0,$$

$$\cdots$$

$$\cdots$$

$$c_{1,p}x_1 + \cdots + c_{n,p}x_n = 0.$$

This is a system of p equations in n unknowns, $x_1, \ldots, x_n$. If p is strictly less than n, then the system will have a free variable, and thus there exists more than the trivial solution. Each such non-trivial solution will correspond to a non-trivial linear dependence relation in the $\mathbf{b_i}$'s. Thus, $p \geq n$.