Linear Algebra
Exam 2 – Fall 2014

November 6, 2014

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators/texts/notes/cell-phones are not permitted – the only permitted item is a pen or pencil. Each problem is worth 5 points unless otherwise indicated.

1. State the Basis Theorem.

2. Let A be a 5×4 matrix. Suppose that the associated linear system $A\mathbf{x} = \mathbf{b}$ has one free variable. Does A have full rank?

3. Find the coordinate vector of $\mathbf{x} = \begin{pmatrix} -1 \\ -6 \end{pmatrix}$ relative to the basis $\mathbf{b_1} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, $\mathbf{b_2} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

4. Let $\mathcal{A} = \{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$ and $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$ be bases for a vector space V, and suppose that $\mathbf{a_1} = 4\mathbf{b_1} - \mathbf{b_2}, \mathbf{a_2} = -\mathbf{b_1} + \mathbf{b_2} + \mathbf{b_3}$, and $\mathbf{a_3} = \mathbf{b_2} - 2\mathbf{b_3}$. Find the change-of-coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$.

5. Compare the following two numbers:

   - the number of multiplications needed in a calculation of the determinant of a 6×6 matrix via co-factor expansion, and
   - the number of different ways for six people to finish a race (where ties are not possible).

6. Give two 2×2 matrices A and B such that $\det(A + B) \neq \det(A) + \det(B)$. Justify your answer.

7. Compute the determinant of the following matrix.

$$A = \begin{bmatrix} 2 & -4 & 10 & 0 \\ 4 & 0 & 8 & -2 \\ 6 & 2 & 0 & 14 \\ 0 & 8 & -4 & 0 \end{bmatrix}$$

8. Each of the following statements are false. Give a counterexample for each in order to demonstrate that the statement is indeed false. Justify each response.

   (a) If the determinant of the 2×2 matrix A is zero, then either the rows are the same, the columns are the same or at least one row or column is all zeros.

   (b) A subset H of the vector space $\mathbb{R}^3$ is a sub space of $\mathbb{R}^3$ if the zero vector is in H.

   (c) If $H = \texttt{Span}\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\} \subseteq \mathbb{R}^4$, then $\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$ is a basis for H.

9. Show that if A is invertible that $\det(A^{-1}) = \frac{1}{\det(A)}$.
Hint: for $n \times n$ *matrices* A *and* B, *we have* $\det(AB) = \det(A)\det(B)$.

10. Theorem 10 of Chapter 4 states: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Let $\mathcal{B}_1$ be a basis of n vectors and $\mathcal{B}_2$ any other basis of V.

When we proved this theorem, there were some questions about why the following statement was true: Since $\mathcal{B}_2$ is a basis and $\mathcal{B}_1$ is linearly independent, $\mathcal{B}_2$ has at least n vectors.

Let us expound on this statement. **Fill in the blanks – 2 points each**.

We need to show that $\mathcal{B}_2$ has at least n vectors. We have that $\mathcal{B}_1 = \{\mathbf{b_1}, \dots, \mathbf{b_n}\}$ is a linearly independent set with n vectors and that $\mathcal{B}_2 = \{\mathbf{v_1}, \dots, \mathbf{v_p}\}$ is a spanning set with, say, p vectors. As $\mathcal{B}_1$ is a linearly independent set, the equation

$$x_1\mathbf{b_1} + \cdots + x_n\mathbf{b_n} = \mathbf{0}$$

has ____. As $\mathcal{B}_2$ is a spanning set, each $\mathbf{b_i}$ is a linear combination of the $\mathbf{v_j}$'s. Thus, we can write the above equation in terms of the $\mathbf{v_j}$'s:

$$x_1(c_{1,1}\mathbf{v_1} + \cdots + c_{1,p}\mathbf{v_p}) + \cdots + x_n(c_{n,1}\mathbf{v_1} + \cdots + c_{n,p}\mathbf{v_p}) = \mathbf{0}.$$

The left side of this equation is a ________________________ of the vectors in $\mathcal{B}_2$, while the right side is $\mathbf{0}$. As $\mathcal{B}_2$ is a linearly independent set, the only solution to this vector equation is the trivial solution. Note however that this equation gives rise to a linear system:

$$c_{1,1}x_1 + \cdots + c_{n,1}x_n = 0,$$
$$\cdots$$
$$\cdots$$
$$c_{1,p}x_1 + \cdots + c_{n,p}x_n = 0.$$

This is a ____________________________________ in n unknowns, $x_1, \dots, x_n$. If ____________________________________, then the system will have a free variable, and thus there exists more than the trivial solution. Each such non-trivial solution will correspond to a non-trivial linear dependence relation in the $\mathbf{b_i}$'s. Thus, $p \geq n$.