Linear Algebra
Exam 2 - Fall 2011

November 10, 2011

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators/texts/notes are not permitted.

1. [6 points] We saw that an invertible square block partitioned matrix of the following form

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

has an inverse of the form

$$\begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

What is the advantage of knowing this? (Be as specific as possible.)

2. [8 points] Find an LU factorization of the following matrix.

$$A = \begin{bmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{bmatrix}$$

3. [8 points] Compute the determinant of the following matrix. Justify your steps.

$$G = \begin{bmatrix} 4 & 8 & 8 & 8 & 5 \\ 6 & 8 & 8 & 8 & 7 \\ 0 & 8 & 8 & 3 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

[2points] Is G invertible? Why?

4. [6 points] The set of vectors $\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}, \mathbf{b_4}\}$ is not a basis for $\mathbb{R}^3$. Find a basis for $\mathbb{R}^3$ that is contained in this set. Justify your answer.

$$\mathbf{b_1} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\mathbf{b_2} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\mathbf{b_3} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\mathbf{b_4} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

5. [5 points] Define **column space** of an $m \times n$ matrix A.

6. [5 points] Prove that the column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

7. [6 points] Given a basis $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}\}$ for $\mathbb{R}^2$, where

$$\mathbf{b_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{b_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and a different basis $\mathcal{C} = \{\mathbf{c_1}, \mathbf{c_2}\}$ for $\mathbb{R}^2$, where

$$\mathbf{c_1} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$\mathbf{c_2} = \begin{bmatrix} 0 \\ 4 \end{bmatrix},$$

we may write the vector

$$\mathbf{x} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

in two different ways: $\mathbf{x} = -2\mathbf{b_1} + 6\mathbf{b_2}$ and $\mathbf{x} = 1\mathbf{c_1} + 1\mathbf{c_2}$. Why doesn't this violate the Unique Representation Theorem?

8. [5 points] With regard to the bases of the previous problem, find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

9. [5 points for each part] Find bases for the row space, the column space and the null space of the matrix A. Note that B is row equivalent to A.

$$A = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

10. [4 points] With regard to the previous problem: what is rank A? what is dim Nul A?