1. [6 points] We saw that an invertible square block partitioned matrix of the following form

\[
\begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix}
\]

has an inverse of the form

\[
\begin{bmatrix}
A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\
0 & A_{22}^{-1}
\end{bmatrix}.
\]

What is the advantage of knowing this? (Be as specific as possible.)
2. [8 points] Find an LU factorization of the following matrix.

\[
A = \begin{bmatrix}
-5 & 0 & 4 \\
10 & 2 & -5 \\
10 & 10 & 16 \\
\end{bmatrix}
\]
3. [8 points] Compute the determinant of the following matrix. Justify your steps.

\[
G = \begin{bmatrix}
4 & 8 & 8 & 8 & 5 \\
6 & 8 & 8 & 8 & 7 \\
0 & 8 & 8 & 3 & 0 \\
0 & 4 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

[2 points] Is $G$ invertible? Why?
4. [6 points] The set of vectors \( \{ \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4 \} \) is not a basis for \( \mathbb{R}^3 \). Find a basis for \( \mathbb{R}^3 \) that is contained in this set. Justify your answer.

\[
\mathbf{b}_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \\
\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \\
\mathbf{b}_3 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \\
\mathbf{b}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]
5. [5 points] Define column space of an $m \times n$ matrix $A$.

6. [5 points] Prove that the column space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^m$. 
7. [6 points] Given a basis $\mathcal{B} = \{b_1, b_2\}$ for $\mathbb{R}^2$, where

$$b_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and a different basis $\mathcal{C} = \{c_1, c_2\}$ for $\mathbb{R}^2$, where

$$c_1 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix},$$

we may write the vector

$$x = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

in two different ways: $x = -2b_1 + 6b_2$ and $x = 1c_1 + 1c_2$. Why doesn’t this violate the Unique Representation Theorem?
8. [5 points] With regard to the bases of the previous problem, find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.
9. [5 points for each part] Find bases for the row space, the column space and the null space of the matrix $A$. Note that $B$ is row equivalent to $A$.

$$A = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

10. [4 points] With regard to the previous problem: what is rank $A$? what is dim $\text{Nul} \ A$?