Name: SOLUTION KEY (Total 55 points, plus 5 more for Pledged Assignment.)

Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators are not permitted.

WARNING: Please do not say that a matrix $A$ is linearly dependent, rather we say that the columns of $A$ are linearly dependent.

The number in square brackets indicates the value of the problem.

1. **Define:** [4] Given vectors $v_1, v_2, \ldots, v_p$ in $\mathbb{R}^n$ then the subset of $\mathbb{R}^n$ spanned by $v_1, v_2, \ldots, v_p$ is . . .

   the collection of all vectors that can be written in the form

   $c_1v_1 + \ldots + c_pv_p$

   with $c_1, \ldots, c_p$ scalars.

   See page 35 of the text.

2. [6] **Define** what it means for a mapping to be **onto**. Give an example of such a mapping.

   A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $b$ in $\mathbb{R}^m$ is the image of at least one $x$ in $\mathbb{R}^n$.

   If $m = n$ then the mapping $x \mapsto I_nx$ is onto, where $I_n$ is the identity matrix. (This mapping is also one-to-one.)

   See page 87 of the text.
3. [4] Assuming that $T$ is a linear transformation, find the standard matrix of $T$, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a vertical shear transformation that maps $e_1$ into $e_1 - 2e_2$, but leaves the vector $e_2$ unchanged.

Using Theorem 10 on page 83, we recall that the standard matrix is $A = [T(e_1 \ldots T(e_n))].$

We find here that $T(e_1) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

And that, $T(e_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

And so,

$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

4. [6] Is the following matrix singular? Why, or why not?

$A = \begin{bmatrix} 1 & 6 \\ -1/2 & 2 \end{bmatrix}$

The matrix is not singular, that is, it is non-singular (invertible) since $(1)(2) - (6)(-1/2) \neq 0$.

Compute the inverse of this matrix and use it to solve the equation $Ax = b$, where

$b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 2/5 & -6/5 \\ 1/10 & 1/5 \end{bmatrix}$

and we obtain that $A^{-1}b$ is equal to

$x = \begin{bmatrix} -16/5 \\ 7/10 \end{bmatrix}$
5. [8] Is the following set of vectors linearly dependent? If not, give a justification. If so, give a linear dependence relation for them.

\[ \mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \]

\[ \mathbf{v}_2 = \begin{bmatrix} -4 \\ -5 \\ -6 \end{bmatrix} \]

\[ \mathbf{v}_3 = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix} \]

If we consider the matrix, \( A \), whose columns are \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \), and perform Gaussian elimination on the augmented matrix corresponding to \( A\mathbf{x} = \mathbf{0} \). We obtain the following reduced echelon form of

\[ A_1 = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

That is, we have \( x_3 \) as a free variable - so that the given column vectors are linearly dependent (again we can invoke the IMT). If we let \( x_3 = 1 \), then \( x_1 = x_2 = 3 \) and we obtain the linear dependence relation of

\[ 3\mathbf{v}_1 + 3\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}. \]

There are infinitely many other possible linear dependence relations.
6. [15 – 3each] True or False: Justify each answer by citing an appropriate definition or theorem. If the statement is false and you can provide a counterexample to demonstrate this then do so. If the statement is false and be can slightly modified so as to make it true then indicate how this may be done.

- If the columns of an \( n \times n \) matrix \( A \) are linearly independent, then the columns of \( A \) span \( \mathbb{R}^n \).

True by the IMT.
- There exists a one-to-one linear transformation mapping \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \).

False. By Theorem 11 (of Chap. 1), \( T \) is 1-1 iff \( T(x) = 0 \) has only the trivial solution. However, the standard matrix of any such transformation is guaranteed a free variable, thus more than the trivial solution.

- If one row in an echelon form of an augmented matrix is \([0 0 0 5 0]\), the the associated linear system is inconsistent.

False, the variable \( x_4 \) could be zero and the system could still be consistent.
- An inconsistent linear system has more than one solution.

False, by definition it has no solutions.
- The codomain of the transformation \( x \mapsto Ax \) is the set of all linear combinations of the columns of \( A \).

False, this is the range.
7. [6] Determine if the following matrices are invertible. Use as few calculations as necessary. Justify your answer.

\[
A_1 = \begin{bmatrix}
1 & 0 & 7 & 5 \\
0 & 1 & 98 & 3 \\
0 & 0 & 33 & -9 \\
0 & 0 & 0 & 11 \\
\end{bmatrix}
\]

The matrix \( A_1 \) has 4 pivots. Thus by the invertible matrix theorem it is invertible.

\[
A_2 = \begin{bmatrix}
1 & 0 & -2 & 11 \\
2 & 1 & -4 & 11 \\
3 & 0 & -6 & 11 \\
4 & 0 & -8 & 11 \\
\end{bmatrix}
\]

For \( A_2 \), the third column is a scalar multiple of the first. Therefore the columns of \( A_2 \) are not linearly independent, which implies by the invertible matrix theorem that \( A_2 \) is NOT invertible.

$$A_2A_1 = \begin{bmatrix} 1 & 0 & -59 & 144 \\ 2 & 1 & -20 & 170 \\ 3 & 0 & -177 & 190 \\ 4 & 0 & -236 & 213 \end{bmatrix}$$

Show in as few calculations possible that this product does not equal $A_1A_2$.

The $(1,1)$ entry in this product is 42, which differs from the $(1,1)$ entry from the above product. Thus, the matrices are not equal.
9. [6] Given $\mathbf{v}_1, \mathbf{v}_2 \neq \mathbf{0}$ and $\mathbf{p}$ in $\mathbb{R}^3$. Further, $\mathbf{v}_1$ is not a scalar multiple of $\mathbf{v}_2$.

- Give a geometric description of the parametric equation $\mathbf{x} = \mathbf{p} + s\mathbf{v}_1 + t\mathbf{v}_2$.

The set of points satisfying this equation is a plane through $\mathbf{p}$, parallel to the plane through the origin containing the vectors $\mathbf{v}_1, \mathbf{v}_2$. A drawing would be appropriate to show.

- Given a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$. Describe the image of the set of vectors satisfying the above parametric equation under $T$.

As $T$ is a linear transformation, we can say that

$$T(\mathbf{p} + s\mathbf{v}_1 + t\mathbf{v}_2) = T(\mathbf{p}) + sT(\mathbf{v}_1) + tT(\mathbf{v}_2).$$

If $T(\mathbf{v}_1), T(\mathbf{v}_1) \neq \mathbf{0}$ then the image of this plane is another plane. It is a line if either, but not both, are $\mathbf{0}$. It is the point given by $T(\mathbf{p})$ if both are $\mathbf{0}$.

This problem generalizes the homework problem from Section 1.8, 25.
10. [5] Given a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Prove the following statement: If $T$ is one-to-one then the equation $T(x) = 0$ has only the trivial solution. Do NOT claim this is true by the Invertible Matrix Theorem. (Note that the IMT would only apply if $n = m$.)

Since $T$ is linear we have that $T(0) = 0$. If $T$ is one-to-one then by the definition the equation $T(x) = 0$ has at most one solution and hence only the trivial solution.

See Theorem 11 on page 88.