Linear Algebra
Exam 1
Fall 2008

October 9, 2008

Name:
Honor Code Statement:

Directions: Complete all problems. Justify all answers/solutions. Calculators are not permitted. There is a two-hour time limit.

1. [5 points]The following transformation is not a linear transformation. Why not? $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $T(x_1, x_2) = (|x_1|, x_2)$.

2. [5 points] **Define** what it means for a mapping to be **one-to-one**. Give an example of a mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$ that is one-to-one.

3. [3 points each] Each of the following statements is false. Amend each statement in as few words as possible, without insertion or deletion of the word "not" (that is, without negating the conclusion), to make a true statement.

   (a) If a system $A\mathbf{x} = \mathbf{b}$ has more than one solution, then the system $A\mathbf{x} = \mathbf{0}$ has a unique solution.

   (b) If a linear system has no free variables, then it has more than one solution.

   (c) If A and B are $n \times n$ invertible matrices, then so is AB and the inverse of AB is $A^{-1}B^{-1}$.

   (d) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if and only if the columns of the standard matrix for T are linearly independent.

4. [8 points] Find the inverse of the following matrix.

$$M = \begin{bmatrix} 4 & 0 & 16 \\ 0 & 1 & 8 \\ 1 & 0 & -5 \end{bmatrix}$$

5. [6 points] Assuming that T is a linear transformation, find the standard matrix of T, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation about the origin through an angle of $\frac{\pi}{4}$ in a *clockwise* direction. .

6. [6 points] Let $A = \begin{bmatrix} 8 & 14 \\ 1 & 2 \end{bmatrix}$.

Compute the inverse of this matrix and use it to solve the equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$.

7. [10 points] Determine the solution set of the following homogeneous linear system. Write the solution set in parametric vector form. Give a geometric description of the solution set.

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 0 \\ x_1 + 4x_2 - 8x_3 &= 0 \\ -3x_1 - 7x_2 + 9x_3 &= 0 \end{aligned}$$

8. [4 points, 3points] Does the following set of vectors span $\mathbb{R}^4$? Give a justification.

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 3 \end{bmatrix}$$

$$\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v_3} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

(a) Is it possible to find vectors $\mathbf{v_4}$ and $\mathbf{v_5}$ in $\mathbb{R}^4$ and add them to the set above and create a linearly independent set? If so, give two such vectors. If not, give a justification.

9. [3 points] If A is the adjacency matrix of a network, what does the entry in row 2, column 3 of A^4 represent?

10. [5 points] Let A be an $n \times n$ matrix. The following statements are not all equivalent. There is one statement which can be removed and we'd be left with an equivalent set of statements. Which one is it? Why?

   (a) A has fewer than n pivots.

   (b) A is singular.

   (c) The columns of A do not span $\mathbb{R}^n$.

   (d) The equation $A\mathbf{x} = \mathbf{0}$ has more than one solution.

   (e) The columns of A do not form a linearly dependent set.

11. [8 points] Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove that if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution then T is one-to-one.