

Linear Algebra

Exam 1

Spring 2026

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Honor Code Statement: I have neither given nor received unauthorized aid on this exam. C.F. Gauss

Directions: Complete all problems. Justify all answers/solutions; notice that some problems have writing limits. Outside sources are not permitted. There is a 90-minute time limit. The exam is proctored with permission of the Dean of the Faculty.

Avg $\frac{47}{60}$

SD 11 points

1. [20 points] Consider the following matrix below.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$$

- (a) Consider the homogeneous equation $Ax = \mathbf{0}$. Write the solution set to this equation in parametric vector form and describe the solution set geometrically.

We perform Gaussian Elimination on the following augmented matrix:

$$\left[A \mid \vec{0} \right] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 1 & 1 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array} \right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 \end{array} \right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{(-1)R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{-2R_2+R_1}$$

Thus, $x_1 = x_3$
 $x_2 = -2x_3$
 x_3 is free.

The solution is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

The solution set is the set of all vectors that are scalar multiples of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, i.e. the line thru the origin in the direction of this vector.

- (b) If possible, write the vector $\mathbf{b} = \begin{bmatrix} -2 \\ -1 \\ -2 \end{bmatrix}$ as a linear combination of the columns of A .

We can perform G.E. on $[A | \vec{b}]$. We'd use the same row operations as above. Doing these just on \vec{b} , we obtain $\begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$, then $\begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix}$ at which point we see a row that is $[0 \ 0 \ 0 | 3]$, indicating an inconsistent system. Thus, this is not possible.

- (c) Is the mapping $\mathbf{x} \mapsto A\mathbf{x}$ an onto mapping? Why or why not?

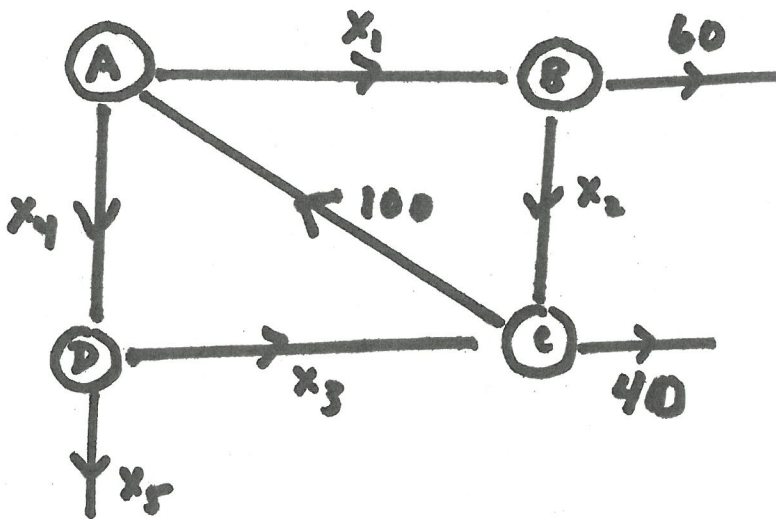
No, it is not onto. In part (b), we have the vector \vec{b} for which there is no pre-image. Or note by the IMT, as there are only 2 pivots the mapping is not onto.

- (d) Give two vectors in the domain of this mapping that map to the same vector in the codomain. Justify your answer.

In part (a) we find many vectors in the domain that map to the zero vector. For example,

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \text{ and } 2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} \text{ both map to } \vec{0}.$$

2. [5 points] Use the figure below to set up a system of equations that models the general flow pattern of the network given. DO NOT solve this system.



We work under the assumption that at each node/vertex the flow in equals the flow out. We state each linear equation generated under this assumption:

	"flow in"	=	"flow out"
A:	100	=	$x_1 + x_4$
B:	x_1	=	$x_2 + 60$
C:	$x_2 + x_3$	=	140
D:	x_4	=	$x_3 + x_5$

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Collecting to one side:

$$\begin{aligned} x_1 + x_4 &= 100 \\ x_1 - x_2 &= 60 \\ x_2 + x_3 &= 140 \end{aligned}$$

3. [5 points] The following set of vectors form a set that is linearly independent. Indicate why, then give an additional vector to add to the set that would make the set of three linearly dependent. Justify your answer.

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

The set doesn't contain the zero vector and w/ only two vectors in it, the only way to be linearly dependent would be if one is a scalar multiple of the other. This is not the case, so the set is lin. ind. Adding any vector that is a linear combination of the given two would make a set of three which will be linearly dependent.

4. [5 points] The following statement is false: "If a set in \mathbb{R}^4 is linearly dependent, then the set contains more than 4 vectors." Show that this is false by giving a counter-example.

Easy! Take $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$. This set is

$$\begin{aligned} \text{For example,} \\ \mathbf{b}_3 &= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

lin dependent since the second is a scalar multiple of the first.

5. [5 points] Show that T is a linear transformation by finding a matrix that implements the mapping.

$$T(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_2 + x_3, x_3 + x_4, x_4)$$

Theorem 10 tells us that we need to know what happens to $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$:

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) & T(\vec{e}_4) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6. [5 points] Determine whether a square matrix A with two identical rows is invertible.
Two sentence writing limit.

In doing Gaussian Elimination, performing a row replacement of $-\text{Row } i + \text{Row } j$ creates a row of zeros in row j . There will be less than n pivots and so by IMT it is not invertible.

7. [5 points] Suppose that A is a 2×2 matrix such that $AA = A^2 = I_2$, that is, A is self-inverse. Show that the following block-partitioned matrix is also self-inverse.

$$M = \begin{bmatrix} A & 0 \\ I_2 & -A \end{bmatrix}$$

We compute $MM = \begin{bmatrix} A & 0 \\ I_2 & -A \end{bmatrix} \begin{bmatrix} A & 0 \\ I_2 & -A \end{bmatrix}$

$$= \begin{bmatrix} A^2 & 0 \\ A-A & A^2 \end{bmatrix} = \begin{bmatrix} A^2 & 0 \\ 0 & A^2 \end{bmatrix}$$

use row-column rule for multiplication.

As $A^2 = I_2$ by the given $M = \begin{bmatrix} I_2 & 0 \\ 0 & I_2 \end{bmatrix} = I_4$

8. [5 points] **Fill-in-the-blank** Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span \mathbb{R}^3 , and let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation such that $T(\mathbf{v}_1) = T(\mathbf{v}_2) = T(\mathbf{v}_3) = \mathbf{0}$. Then for any \mathbf{x} in \mathbb{R}^3 , there are constants c_1, c_2, c_3 such that $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ because $\vec{v}_1, \vec{v}_2, \vec{v}_3$ span. Then we can write

$$T(\mathbf{x}) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3)$$

since T preserves vector addition + scalar mult. Now since $T(\mathbf{v}_1) = T(\mathbf{v}_2) = T(\mathbf{v}_3) = \mathbf{0}$, we have that $T(\mathbf{x})$ equals $\vec{0}$. In this way, we might name/describe T as the zero/trivial transformation.