Linear Algebra
Exam 1
Spring 2017

March 16, 2017

Name:
Honor Code Statement:

Avg $\frac{46.8}{60}$

Additional Statement: I have not observed another violating the Honor Code.

Directions: Complete all problems. Justify all answers/solutions. Calculators, notes or texts are not permitted. Cell phones should not be used at any time (even to check the time) - please put them away! There is a two-hour time limit.

1. [5points] Suppose T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ that projects onto the x_1, x_2-plane, that is, $T(x_1, x_2, x_3) = (x_1, x_2, 0)$. Give the standard matrix of the transformation. Also, give a counterexample to the following statement: T is one-to-one.

By Theorem 10 of Chapter 1, the standard matrix of T is given by $A = \left[T(\vec{e}_1) \ T(\vec{e}_2) \ T(\vec{e}_3) \right]$. We find these column vectors: $T(\vec{e}_1) = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $T(\vec{e}_2) = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $T(\vec{e}_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Thus, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

T is not one-to-one if we can find some $\vec{b} \in \mathbb{R}^3$ which is the image of more than one $\vec{x} \in \mathbb{R}^3$.

Say $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, then for $\vec{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 17 \end{bmatrix}$ we have $T(\vec{x}) = T(\vec{y}) = {}^1\vec{b}$.

2. [3 points each] Each of the following statements is false. Amend each statement in as few words or symbols as possible, without insertion or deletion of the word "not" (that is, without negating the conclusion), to make a true statement or explain why it's false, which can be done by pointing to a theorem or giving a counter-example.

(a) If every column of an augmented matrix contains a pivot, then the corresponding system is consistent.

- Change ~~column~~ augmented to coefficient.
- OR change column to row and augmented to coefficient (via Theorem of Chapter 1
- OR insert "If every column ... except the right-most."

(b) If the coefficient matrix A has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent.

Replace inconsistent with consistent.

(c) If $\mathbf{x}$ is a nontrivial solution of $A\mathbf{x} = \mathbf{0}$, then every entry in $\mathbf{x}$ is nonzero.

Replace every by some.

(d) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and if $\mathbf{c}$ is in $\mathbb{R}^m$, then a uniqueness question is "Is $\mathbf{c}$ in the range of T?"

Replace uniqueness by existence.

(e) Suppose that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are in $\mathbb{R}^5$, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$ and $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Add "and the set does not contain the zero vector".
(see Theorem 7? of Ch. 1)

Note: Many students incorrectly thought that $\vec{v}_3$ might be a multiple of $\vec{v}_1$ or $\vec{v}_2$. However, if $\vec{v}_3$ is not a linear combination of $\vec{v}_1$ and $\vec{v}_2$, it cannot be a multiple of either.

4. [10 points] Suppose that $T : \mathbb{R}^4 \to \mathbb{R}^3$ is an onto linear transformation. Describe the *four* different possible echelon forms of the standard matrix for T, where $\blacksquare$ corresponds to a pivot entry, and $*$ corresponds to any real number.

need I say 0 corresponds to zero?

As T is onto the columns of the standard matrix span $\mathbb{R}^3$ — this is by Theorem 12. So, by Theorem 4, A has a pivot in each row. Thus, the possibilities.

(1) $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}$ (2) $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$

(3) $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$ (4) $\begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$

True or False: T can be one-to-one.

False, T cannot be one-to-one.

Examining the possible echelon forms above, we see that in each 4 case there is a free variable, in the equation $A\vec{x} = \vec{b}$, implying that the map is many-to-one.

3. [5 points] Limestone, $CaCO_3$, neutralizes the acid, H_3O, in acid rain by the following unbalanced equation:

$$H_3O + CaCO_3 \to H_2O + Ca + CO_2.$$

Write a vector equation that demonstrates how one would balance this equation. (You need **not** solve this equation.)

Let a vector in $\mathbb{R}^4$ list (in order) the number of atoms of hydrogen (H), oxygen (O), calcium (Ca) and carbon (C).

Then the following correspondences hold:

$H_3O : \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $CaCO_3 : \begin{bmatrix} 0 \\ 3 \\ 1 \\ 1 \end{bmatrix}$ $H_2O : \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $Ca : \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

and $CO_2 : \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$.

Then we may represent the chemical equation

$$x_1 H_3O + x_2 CaCO_3 \to x_3 H_2O + x_4 Ca + x_5 CO_2$$

as the following vector equation

$$x_1 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 1 \\ 1 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

5. [5 points] Show that the following matrix does not have an inverse.

$$M = \begin{bmatrix} 2 & 3 & -1 \\ -8 & -7 & 6 \\ 6 & -1 & -7 \end{bmatrix}$$

One way to do this is to row reduce M and count the number of pivots. By the Invertible Matrix Theorem, M will not have an inverse iff the number of pivots is less than 3.

$$M \sim \underset{4R1+R2}{\begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 6 & -1 & -7 \end{bmatrix}} \sim \underset{-3R1+R3}{\begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & -10 & -4 \end{bmatrix}} \sim \underset{2R2+R3}{\begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$$

We now see the number of pivots is two, and so M is not invertible.

6. [5 points] Define what it means for a transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ to be **onto**.

A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto if each $\vec{b} \in \mathbb{R}^m$ is the image of at least one $\vec{x}$ in $\mathbb{R}^n$.

7. [10 points total] **Construct an example:** In each of the following prompts give an example of the desired object(s) **and** *demonstrate/prove* that these objects have the desired property.

(a) A 2×2 matrix A and a 2×2 matrix B which do not commute.

Many possibilities. Here's one such: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Thus, $AB \neq BA$, i.e. they do not commute.

(b) An 3×3 elementary matrix whose inverse has the entry 1 in row 1, column 2.

The matrix $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix

with inverse $E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Another possibility is $E_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which has inverse $E_2^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) An invertible 4×4 matrix for which precisely one calculation (e.g. elementary row operation, matrix multiplication, etc.) is needed to demonstrate its invertibility.

One possibility is $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

Then the elementary row operation $-R3 + R4$ shows that A has echelon form $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. We now see 4 pivots and so by the Invertible Matrix Theorem A is invertible.

8. [5 points] Suppose $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set in $\mathbb{R}^n$. Show that $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2\}$ is also a linearly independent set.

Suppose that there are constants c_1 and c_2 satisfying

$$c_1 \vec{v}_1 + c_2 (\vec{v}_1 + \vec{v}_2) = \vec{0}. \qquad (1)$$

Then we have $(c_1 + c_2)\vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$.

As $\{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set, by definition we must have $c_2 = 0$ and $c_1 + c_2 = 0$.

These imply that c_1 must also equal 0.

Thus the only solution to equation (1) is the trivial solution, in other words $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2\}$ is a linearly independent set.