Linear Algebra
Exam 1
Spring 2014

March 13, 2014

Name: Solution Key
Honor Code Statement: I have neither given nor received unauthorized aid on this examination.

Total # of points 65

Directions: Complete all problems. Fill-in-the-blank problems are worth 2 points each and all others are worth 5 points each. Justify all answers/solutions. Calculators, notes or texts are not permitted. There is a two-hour time limit.

1. Use Gaussian Elimination to solve the following linear system. Identify pivot columns. Identify free variables. Express your answer in parametric vector form.

$$x_1 - 3x_2 - 5x_3 = 0$$
$$x_1 - 2x_2 - 4x_3 = 3$$

Consider the associated augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -3 & -5 & 0 \\ 1 & -2 & -4 & 3 \end{array}\right]$$

As the (1,1)-entry is non-zero, the first column is a pivot column. So we clear below this entry.

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -5 & 0 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

The (2,2)-entry is non-zero, so the 2nd column is a pivot column. As the # of variables exceeds the # of rows, the 3rd column corresponds to the variable x_3 being a free variable. We now clear "above".

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 9 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

Thus, $x_1 = 2x_3 + 9$, $x_2 = -x_3 + 3$, x_3 free. In parametric vector form,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 + 9 \\ -x_3 + 3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 9 \\ 3 \\ 0 \end{bmatrix}$$

2. Compute the following product: $A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 1 & 0 & 5 \\ -1 & 1 & -1 & 11 \\ 2 & 0 & 2 & 6 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$.

$$A\vec{x} = 1\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + -1\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 11 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 \\ -1+2+1 \\ 2-2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

3. In regards to the matrix in the previous problem, what does the size of A tell you about its invertibility?

It is not invertible since number of columns exceeds number of rows.
Our definition of invertible only applies to square matrices.
(You cannot apply IMT since that theorem applies to square matrices.)

4. Find the inverse of the following elementary matrix.

$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ We apply the algorithm which is yielded by Theorem 7 of Section 2.3.

$$\left[M \mid I_3 \right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \underset{-2R1+R2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

Thus $M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

5. Give 5 statements that are equivalent to the following: an $n \times n$ matrix A is *not* an invertible matrix.

Here are several that we know at this point.

(a) *A is not row equivalent to the $n \times n$ identity matrix.*

(b) *A does not have n pivot positions.*

(c) *The equation $A\mathbf{x} = \mathbf{0}$ has more than the trivial solution.*

(d) *The columns of A form a linearly dependent set.*

(e) *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not one-to-one.*

(f) *The equation $A\mathbf{x} = \mathbf{b}$ has no solution or more than one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.*

(g) *The columns of A do not span $\mathbb{R}^n$.*

(h) *The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ does not map $\mathbb{R}^n$ onto $\mathbb{R}^n$.*

(i) *There is no $n \times n$ matrix C such that $CA = I_n$.*

(j) *There is no $n \times n$ matrix D such that $AD = I_n$.*

(k) *A^T is not an invertible matrix.*

Be sure to avoid use of the word it. Refer to the columns of A when you need to, not just A.

6. Define the *subset of* $\mathbb{R}^n$ *spanned by* $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$.

This is the set of all linear combinations of the vectors $\mathbf{v_1}, \ldots, \mathbf{v_p}$. That is, $\{c_1\mathbf{v_1} + \ldots + c_p\mathbf{v_p} : c_1, \ldots, c_p \in \mathbb{R}\}$.

7. Geometrically, for a non-zero vector $\mathbf{v}$, Span$\{\mathbf{v}\}$ is a line through the origin in the direction of $\vec{v}$.
Geometrically, for the zero vector $\mathbf{0}$, Span$\{\mathbf{0}\}$ is simply the origin.

(The more specificity you gave, the more points earned.)

8. The following transformation, which maps vectors from $\mathbb{R}^2$ to vectors in $\mathbb{R}^2$, is not onto: $\mathbf{x} \mapsto A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Give a vector in the codomain that has no pre-image to demonstrate that this is so.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be a vector in the domain $\mathbb{R}^2$.

Then the image of $\vec{x}$ under the mapping is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$.

Thus, the range of this mapping is the x_2-axis.

So if we choose any vector with a non-zero first entry, then this vector will have no pre-image.

For example, let $\vec{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

Gaussian elimination will <u>also</u> answer the question.

Consider $A\vec{x} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, we can show this is inconsistent:

$\left[\begin{array}{cc|c} 0 & 0 & 5 \\ 0 & 1 & 0 \end{array}\right]$, i.e. $\nexists\ \vec{x}$ that satisfies the equation.

9. The following transformation, which maps vectors from $\mathbb{R}^2$ to vectors in $\mathbb{R}^2$, is not one-to-one: $\mathbf{x} \mapsto A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

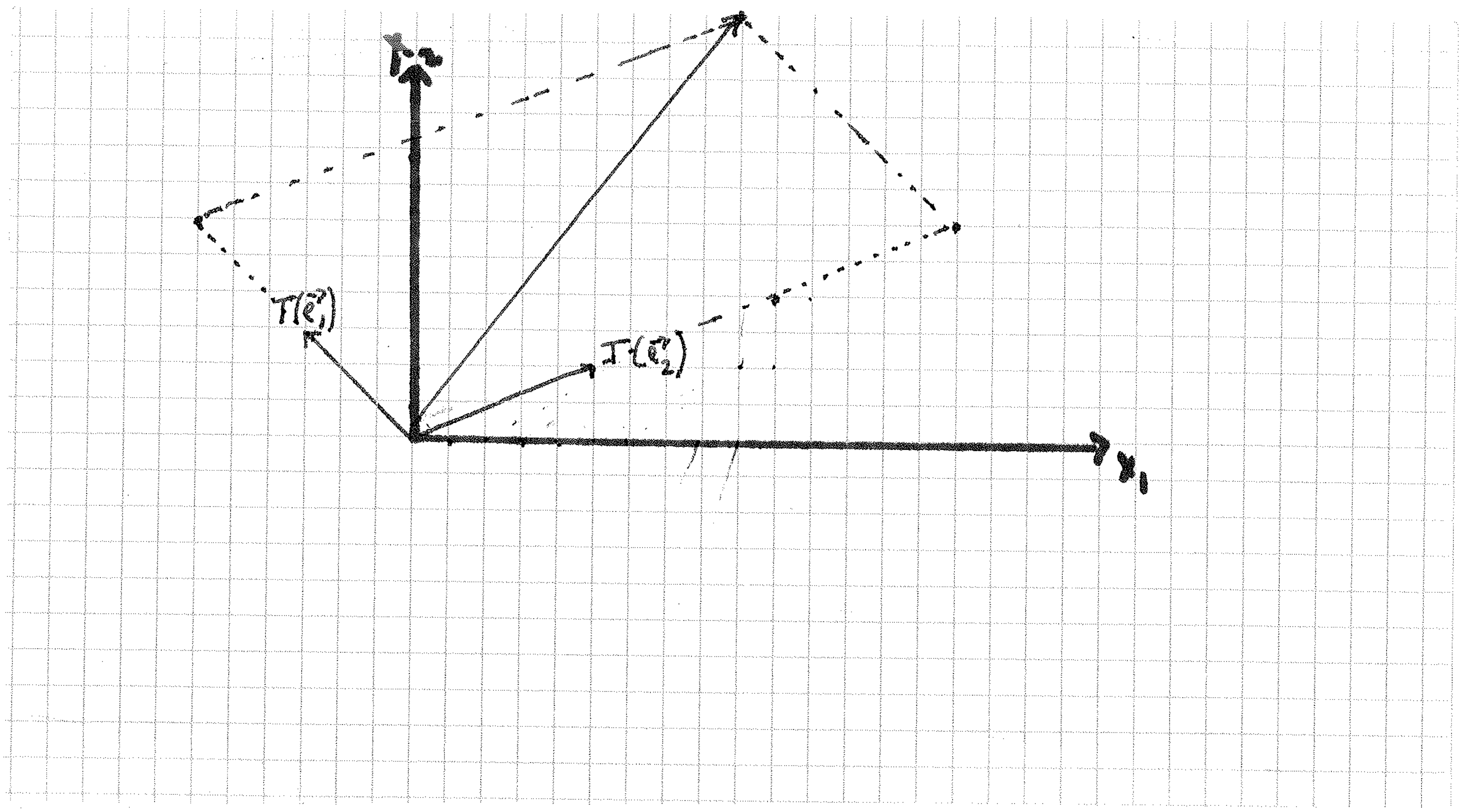
Give a vector in the codomain and two of its pre-images to demonstrate that this is so.

Let $\vec{b} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$ then two of its pre-images are

$\begin{bmatrix} 8 \\ 8 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ 64 \end{bmatrix}$ since $A\begin{bmatrix} 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$ and $A\begin{bmatrix} 8 \\ 64 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$.

10. In regards to the previous problem: (fill-in-the-blank) we know that this transformation is not one-to-one by examining the columns of A – this set of columns is linearly dependent.

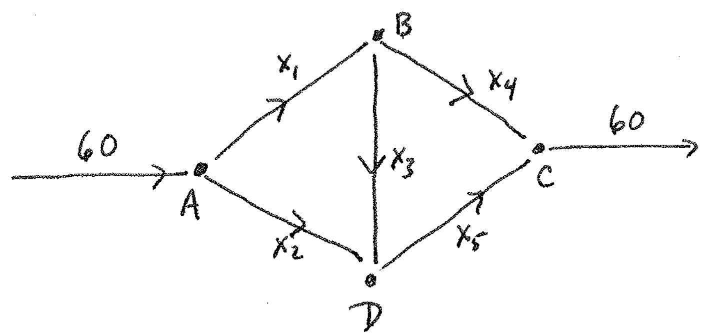
11. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. The illustration below shows $T(\mathbf{e_1})$ and $T(\mathbf{e_2})$. Accurately sketch $T(2, 3)$.



The illustration shows that $T(\vec{e}_1) = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$ and $T(\vec{e}_2) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. Thus the standard matrix of the transformation is $A = \begin{bmatrix} -3 & 5 \\ 3 & 2 \end{bmatrix}$. Thus $T(2,3)$ is given by $\begin{bmatrix} -3 & 5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$.

Alternately, as T is linear, $T(2,3) = 2T(1,0) + 3T(0,1)$ and so we can take 2 copies of $T(\vec{e}_1)$ and 3 copies of $T(\vec{e}_2)$ and add these together.

12. Consider the following network flow problem as illustrated below. *Assuming* that there is a solution (i.e. you needn't do any Gaussian elimination) to the flow problem, is there another? Why?



The network flow problem gives rise to a system of equations. The number of equations corresponds to the number of nodes, which is 4. The number of unknowns is 5. As the number of unknowns exceeds the number of equations, there is a free variable. Since we're assuming there is a solution, there exists another.

13. The following transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ is not a linear transformation: $T(\mathbf{x}) = 3\mathbf{x} + \mathbf{v}$, where $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Give a brief argument to support the claim that T is not a linear transformation.

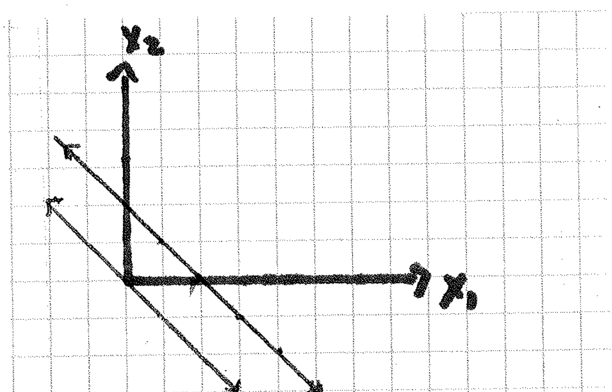
If T is a linear transformation, then $T(\vec{0}) = \vec{0}$.

For this transformation, we see that $T(\vec{0}) = \vec{v} \neq \vec{0}$.

Thus T cannot be a linear transformation.

One could also show that $T(\vec{a} + \vec{b}) \neq T(\vec{a}) + T(\vec{b})$ for some pair of vectors $\vec{a}, \vec{b}$. Or one could show $T(c\vec{x}) \neq cT(\vec{x})$ for some vector $\vec{x}$ and scalar c.

14. The solution set of the matrix equation $A\mathbf{x} = \mathbf{0}$ is illustrated below, where A is a 2×2 matrix. Suppose that for some non-zero $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has the solution $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Draw the entire solution set for $A\mathbf{x} = \mathbf{b}$.



← solution set to $A\vec{x} = 0$

The solution set for $A\vec{x} = \vec{b}$ passes through $(1,1)$ and is parallel to the solution set for $A\vec{x} = \vec{0}$. See Theorem 6 of Section 1.5.

15. In the above problem, both the domain and codomain of the transformation (of multiplication by A) are $\mathbb{R}^2$. But, does the illustration above represent the domain or the codomain? It represents the ____domain.____.