Linear Algebra
Exam 1
Spring 2013

March 14, 2013

Total Points 60

Name: Key
Honor Code Statement: I have neither given nor received any unauthorized aid.

Additional Statement: I have not observed another violating the Honor Code.

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Directions: Complete all problems. Justify all answers/solutions. Calculators, notes or texts are not permitted. There is a two-hour time limit.

1. [3 points each] Each of the following statements is false. Amend each statement in as few words or symbols as possible, without insertion or deletion of the word "not" (that is, without negating the conclusion), to make a true statement or explain why it's false.

underlining indicates where change occurred

   (a) A homogeneous system of equations can be inconsistent.

   Possible alterations (1) A nonhomogeneous system.... (2) is always consistent, since it has the trivial solution

   (b) When **u** and **v** are nonzero vectors, $Span\{\mathbf{u}, \mathbf{v}\}$ contains only the line through **u** and the origin, and the line through **v** and the origin.

   Possible alterations (1) delete "only", (2) insert "parallel" vectors and delete "and the line..."

   (c) If a set in $\mathbb{R}^n$ is linearly dependent, then the set contains more than n vectors.

   Possible alterations (1) If a set in $\mathbb{R}^n$ contains more than n vectors, then it is linearly dependent.

   (2) If a set in $\mathbb{R}^n$ is linearly independent, then the set contains at most n vectors.

   (d) A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every vector $\mathbf{x}$ in $\mathbb{R}^n$ maps onto some vector in $\mathbb{R}^m$.

Possible alterations:

(1) .... if every vector $\vec{x}$ in $\mathbb{R}^m$ has a pre-image in $\mathbb{R}^n$.

(2) ... is defined if every ....

2. [8 points] Find the inverse of the following matrix.

$$M = \begin{bmatrix} -2 & -1 & -2 \\ -1 & 1 & -1 \\ 1 & -2 & 3 \end{bmatrix}$$

We follow/use the algorithm outlined on page 108 of the text. So we augment M by I and perform Gaussian Elimination.

$$\left[\begin{array}{ccc|ccc} -2 & -1 & -2 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 1 & -2 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1/2 & 1 & -1/2 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 1 & -2 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1/2 & 1 & -1/2 & 0 & 0 \\ 0 & 3/2 & 0 & -1/2 & 1 & 0 \\ 0 & -5/2 & 2 & 1/2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1/2 & 1 & -1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/3 & 2/3 & 0 \\ 0 & -5/2 & 2 & 1/2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1/2 & 1 & -1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/3 & 2/3 & 0 \\ 0 & 0 & 2 & -1/3 & 5/3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1/2 & 1 & -1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & -1/6 & 5/6 & 1/2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1/2 & 0 & -1/3 & -5/6 & -1/2 \\ 0 & 1 & 0 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & -1/6 & 5/6 & 1/2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/6 & -7/6 & -1/2 \\ 0 & 1 & 0 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & -1/6 & 5/6 & 1/2 \end{array}\right]$$

Thus, $M = \begin{bmatrix} -1/6 & -7/6 & -1/2 \\ -1/3 & 2/3 & 0 \\ -1/6 & 5/6 & 1/2 \end{bmatrix}$

We double-check for arithmetic errors by computing $M \cdot M^{-1}$

$$\begin{bmatrix} -2 & -1 & -2 \\ -1 & 1 & -1 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} -1/6 & -7/6 & -1/2 \\ -1/3 & 2/3 & 0 \\ -1/6 & 5/6 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. [6 points] Using the matrix M from the previous problem, re-write the matrix equation $M\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, as a system of linear equations *and* as a vector equation.

As a system of linear equations, we have

$$\begin{aligned} -2x_1 + -1x_2 - 2x_3 &= 1 \\ -x_1 + x_2 - x_3 &= 1 \\ x_1 - 2x_2 + 3x_3 &= 2. \end{aligned}$$

As a vector equation, we have

$$x_1 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

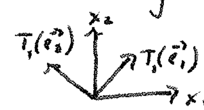
* Note: You weren't asked to solve for $\vec{x}$.

4. [3 points] State what it means for a set of vectors to *span* $\mathbb{R}^3$.

A set of vectors S spans $\mathbb{R}^3$ if each $\vec{x} \in \mathbb{R}^3$ can be written as a linear combination of the elements in S.

5. [10 points]The linear transformation $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ rotates points through $\frac{\pi}{4}$ radians in a counterclockwise fashion. What is the matrix transformation that achieves this?

According to Theorem 10 of Section 1.9, the standard matrix is $A_1 = [T_1(\vec{e_1}) \; T_1(\vec{e_2})]$.
So we determine $T_1(\vec{e_1})$ and $T_1(\vec{e_2})$. Here is a figure to help illustrate – it shows $T_1(\vec{e_1})$ and $T_1(\vec{e_2})$.



By right-triangle geometry, $T_1(\vec{e_1}) = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $T_1(\vec{e_2}) = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Thus the matrix transformation is $A_1 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

The linear transformation $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ maps $\mathbf{e}_1$ to itself and $\mathbf{e}_2$ to the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. What is the matrix transformation that achieves this?

Here we see that $T_2(\vec{e_1}) = \vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T_2(\vec{e_2}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Thus, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Using matrix algebra, show that ~~the~~ $T_1(T_2(\mathbf{x})) \neq T_2(T_1(\mathbf{x}))$ for some $\mathbf{x}$ in $\mathbb{R}^2$.

To do this we show that $A_1 A_2 \neq A_2 A_1$, since matrix multiplication corresponds to function composition.

$$A_1 A_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 2/\sqrt{2} \end{bmatrix}$$

$$A_2 A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

6. [4 points] Find the parametric equation of the line through $\mathbf{a} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ and parallel to $\mathbf{b} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$.

The line through $\vec{a}$ parallel to $\vec{b}$ can be written as $\vec{x} = \vec{a} + t\vec{b}$, where t represents a parameter. So,

$$\vec{x} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 3 \end{bmatrix}, \quad \text{or} \quad \begin{aligned} x_1 &= -2 - 5t \\ x_2 &= 3t \end{aligned}$$

7. [4 points] To reduce a matrix to reduced echelon form, there are two phases, the forward phase and the backward phase. If you had to do one of these phases (by hand) for a 25×25 matrix and your best friend had to do the other, which would you choose and why?

As the forward phase generally takes much longer than the backward phase, I would choose the backward phase and leave my "friend" with the majority of the work to be done. The forward phase takes about 25^3 flops, whereas the backwards phase takes about 25^2 flops. (See the numerical note on page 20.)

8. [8 points] Let $T : \mathbb{R}^n \to \mathbb{R}^m$ have the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, with A an $m \times n$ matrix and $\mathbf{b}$ in $\mathbb{R}^m$. (Such a transformation is called an *affine transformation.*) Suppose that $\mathbf{b} = \mathbf{e}_1$, is T linear? Justify your answer.

T is not linear.

We will show that this mapping does not take the zero vector to the zero vector, a property that must hold for a transformation to be linear.

$$T(\vec{0}) = A(\vec{0}) + \vec{e}_1 = \vec{0} + \vec{e}_1 = \vec{e}_1 \neq \vec{0}.$$

So, the transformation is not linear.

Alternately, we could show that vector addition or scalar multiplication are not preserved.

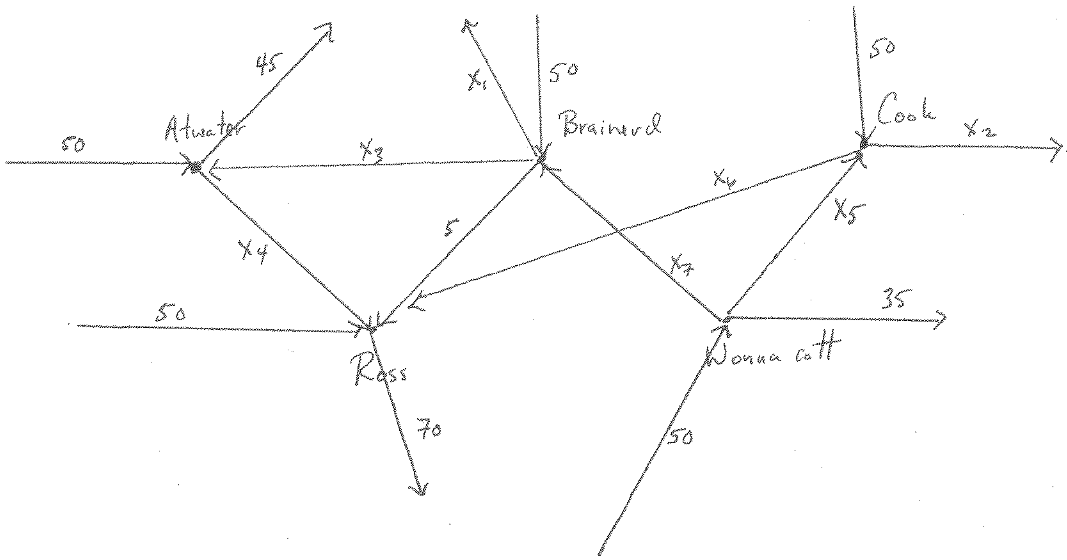
① Showing vector addition is not preserved:

$$\left.\begin{aligned} T(\vec{u}+\vec{v}) &= A(\vec{u}+\vec{v}) + \vec{e}_1 = A\vec{u} + A\vec{v} + \vec{e}_1 \\ T(\vec{u}) + T(\vec{v}) &= A\vec{u} + \vec{e}_1 + A\vec{v} + \vec{e}_1 = A\vec{u} + A\vec{v} + 2\vec{e}_1 \end{aligned}\right\} \text{these are not equal.}$$

② Showing scalar multiplication is not preserved:

$$\left.\begin{aligned} cT(\vec{u}) &= c(A\vec{u} + \vec{e}_1) = Ac\vec{u} + c\vec{e}_1 \\ T(c\vec{u}) &= Ac\vec{u} + \vec{e}_1 \end{aligned}\right\} \text{these are not equal for any } c \neq 1.$$

9. [5 points] Given the flow pattern of the network below and without solving the associated linear system, if the flow pattern has a solution, is there a second solution? Justify your answer.



If you write down the associated linear system (where we have assumed conservation of flow), then you should note that there are 5 equations and 7 unknowns. As there is a solution, the system is consistent and there are at least two free variables. With free variables there will be an infinite number of solutions (though note that we may not have each entry of a given solution as positive).